Vertex Colourings

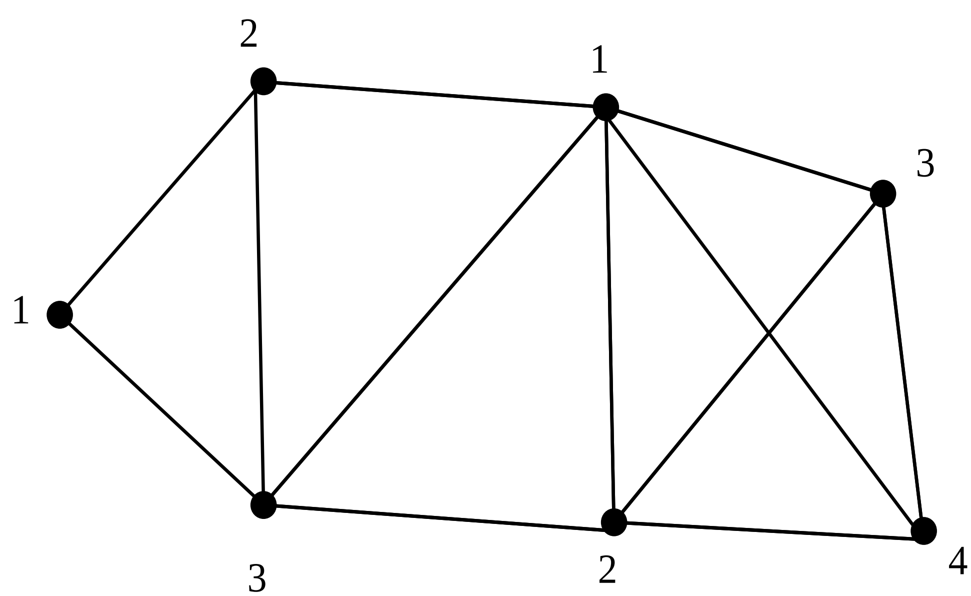
We assume in this chapter that G is simple.

A $k - colouring$ of (the vertices of) G is a mapping

$$c : \; V \rightarrow \{1, 2, \ldots, k\}.$$

$c(v)$ is the colour of vertex v.

$K_i = \{v \in V \; : \; c(v) = i\}$ is the set of vertices with colour i.



c is *proper* if $K_1, K_2, \ldots, K_k$ are independent sets i.e. adjacent vertices v, w have $c(v) \neq c(w)$.

G is $k-colourable$ if it has a proper k- colouring.
A graph is k-colourable iff it is k-partite.
The *Chromatic Number*

$$\chi(G) = \text{min}\{k : \ G \text{ is } k\text{-colourable}\}.$$

Lemma 1

$$\chi(G) \geq \text{max}\{cl(G), \nu/\alpha(G)\}$$

where $cl(G)$ is the size of the largest clique n G.

Proof If C is a clique of G then every vertex of C must have a different colour in a proper colouring of G.

If $K_1, K_2, \ldots, K_k$ defines a proper k-colouring then

$$\nu = \sum_{i=1}^{k} |K_i| \leq k\alpha(G).$$

$\square$

Greedy Colouring Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$ and $V_i = \{v_1, v_2, \ldots, v_i\}$ for $i = 1, 2, \ldots, n$.

begin

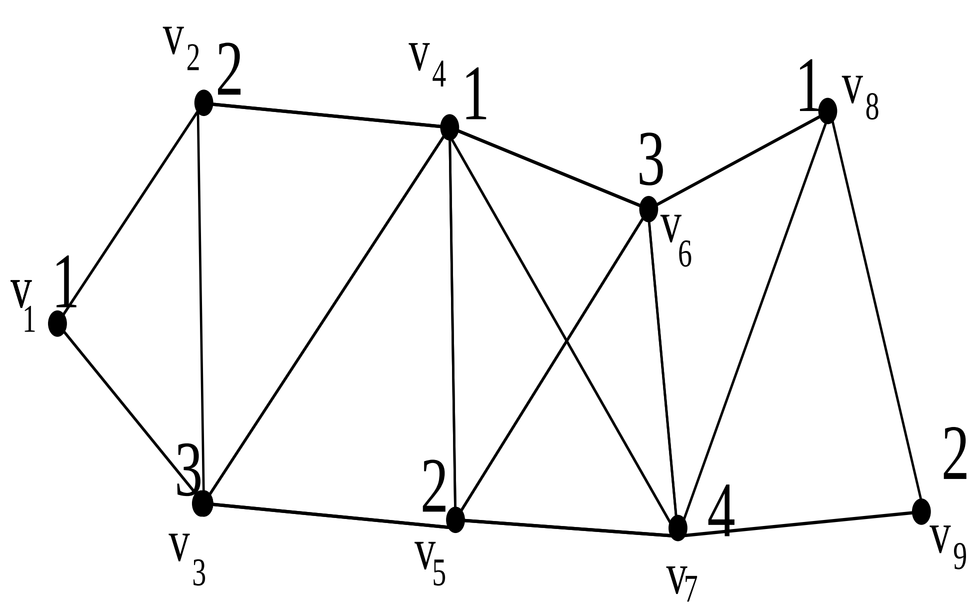
for $i = 1$ **to** n **do**

begin

$c(v_i) := \min\{j : \nexists w \in N_G(v_i) \cap V_{i-1}$ with $c(w) = j\}$

end

end



Theorem 1

$$\chi(G) \leq \Delta(G) + 1.$$

The Greedy Colouring algorithm produces a proper k-colouring for some $k \leq \Delta + 1$ where

$$k \leq 1 + \max_i |N_G(v_i) \cap V_{i-1}|. \tag{1}$$

(a) The colouring is proper: Suppose $v_r v_s \in E$ and $r < s$. $c(v_r) \neq c(v_s)$ since $c(v_s)$ is the lowest numbered colour that is not used by a neighbour of v_s in $\{v_1, v_2, \ldots, v_{s-1}\}$,

(b) At most $\Delta + 1$ colours are used: $|N_G(v_i)| \leq \Delta$ and so the minimum above is never more than $\Delta + 1$.

$\square$

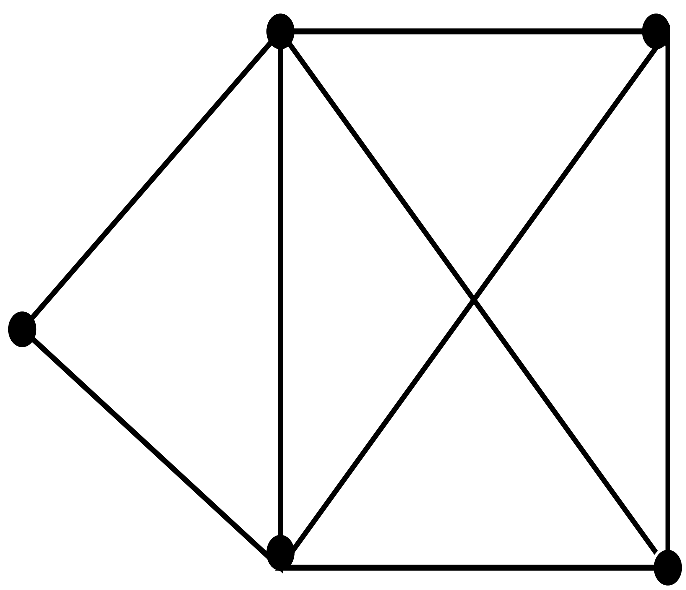
If G is a complete graph or an odd cycle then $\chi(G) = \Delta + 1$.

Colouring Number

Let

$$\delta^*(G) = \max_{S \subseteq V} \delta(G[S])$$

(the maximum over the vertex induced subgraphs of their minimum degrees.)



G

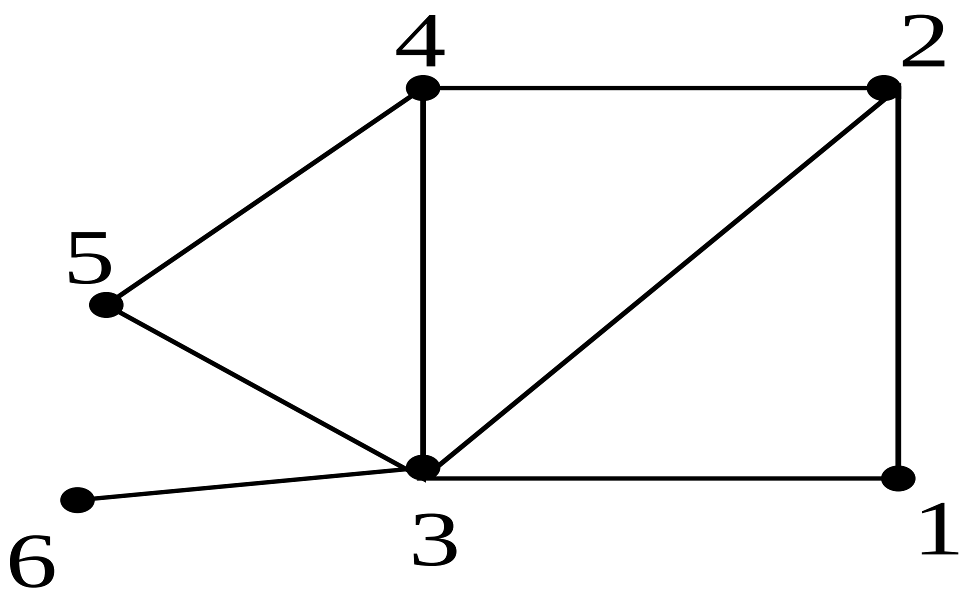
$\delta(G) = 2$ and $\delta^*(G) = 3$.

Theorem 2

$$\chi(G) \leq \delta^*(G) + 1.$$

Proof Let $V = \{v_1, v_2, \ldots, v_n\}$ where

v_i is a minimum degree vertex of $G[V_i]$.



Run the greedy colouring algorithm with this vertex order.

$$|N_G(v_i) \cap V_{i-1}| = \delta(G[V_i]) \leq \delta^*.$$

The theorem follows from (1). $\square$

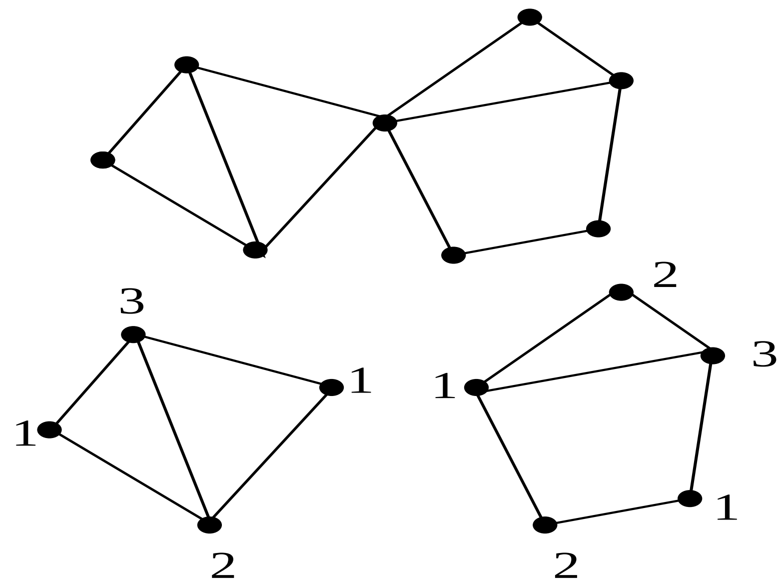
Brook's Theorem

Theorem 3 *If G is a connected graph which is not a complete graph or an odd cycle then $\chi(G) \leq \Delta(G)$.*

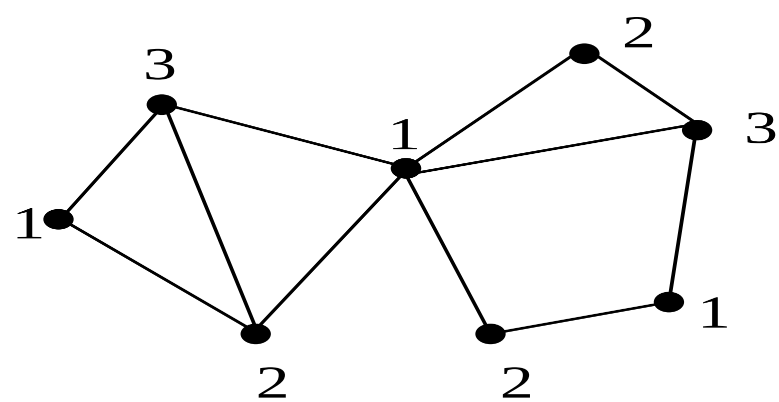
Proof We shall prove this by induction on the number of vertices in G.

Assume that G is connected but not a complete graph or an odd cycle.

If G has a cutpoint v let $G - v$ have components $C_1, C_2, \ldots, C_p$ and let $W_i = C_i + v$ for $i = 1, 2, \ldots, p$. Let $k_i = \chi(G[W_i])$ and properly k_i-colour the vertices of each W_i so that v has colour 1 in each.



This induces a proper k-colouring of G where $k = \max\{k_1, k_2, \ldots, k_p\}$.

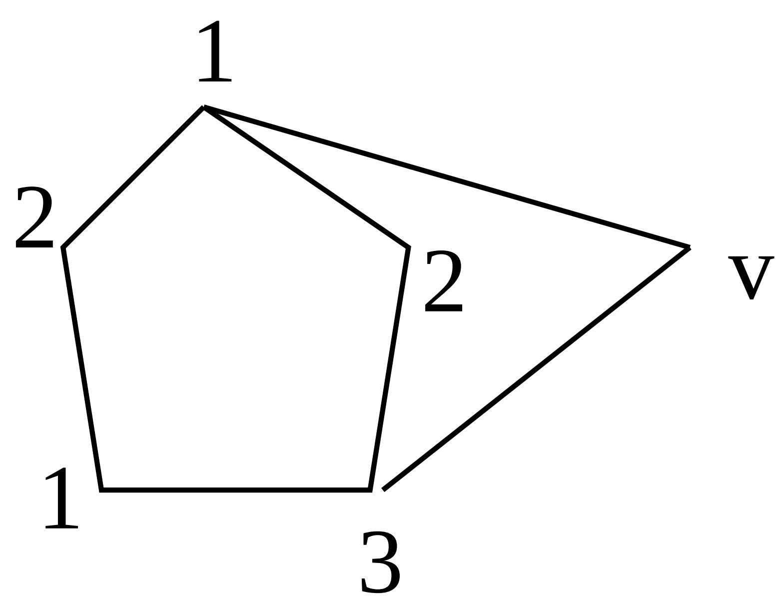


We argue that $k \leq \Delta$. If say $k_1 = \Delta + 1$ then (by induction) either W_1 is an odd cycle or a complete graph on k_1 vertices..

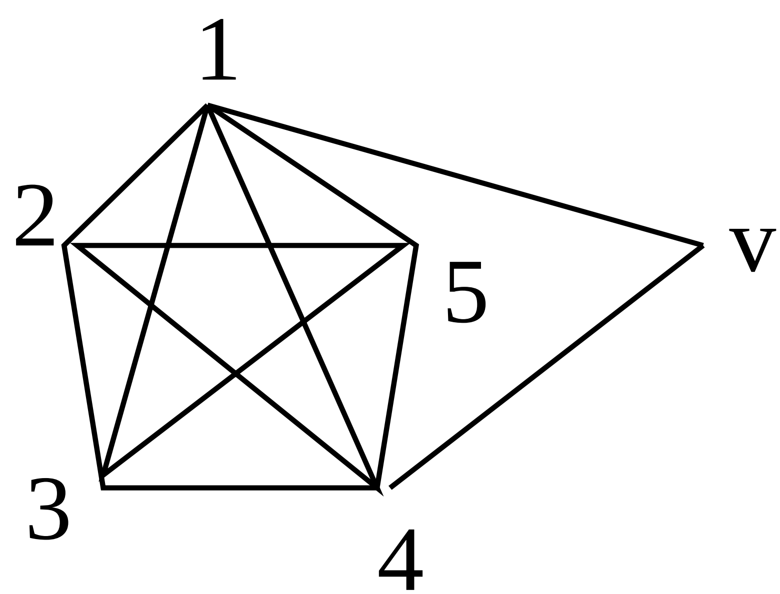
If W_1 is an odd cycle then $k_1 = 3$ and $\Delta = 2$ but now $d_G(v) \geq 3$ — contradiction.

If W_1 is a complete graph on k_1 vertices then $\Delta \geq d_G(v) \geq k_1$ — contradiction.

Suppose next that G contains a vertex v with $d_G(v) \leq \Delta - 1$. Let $H = G - v$.
If H is an odd cycle then $\Delta(G) = 3$. We can 3-colour H and then colour v with a colour not used by one of its ≤ 2 neighbours. Thus $\chi(G) = 3$ as required.



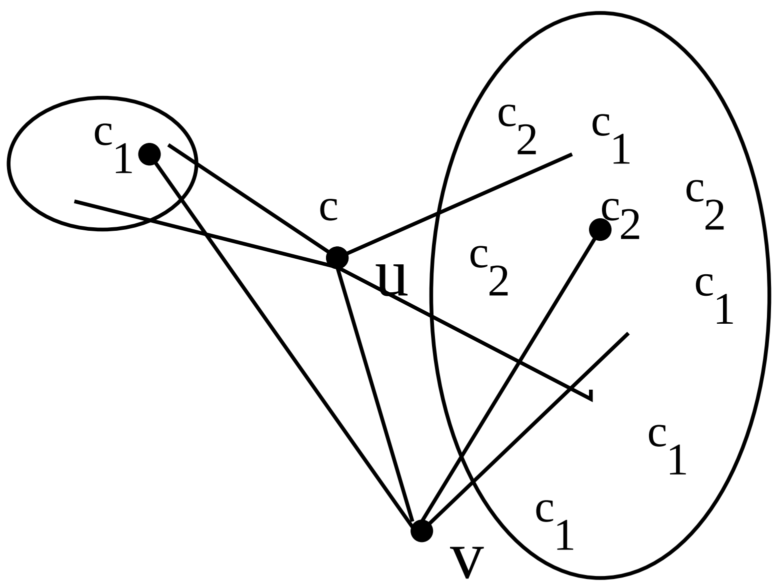
If H is a k-clique then $\Delta(G) = k$. We k-colour H and extend the colouring to v as v has less than k neighbours in H.



If H is is neither a clique or an odd cycle then we can Δ-colour it. We can extend this colouring to v by using one of the colours not used so far in $N_G(v)$.

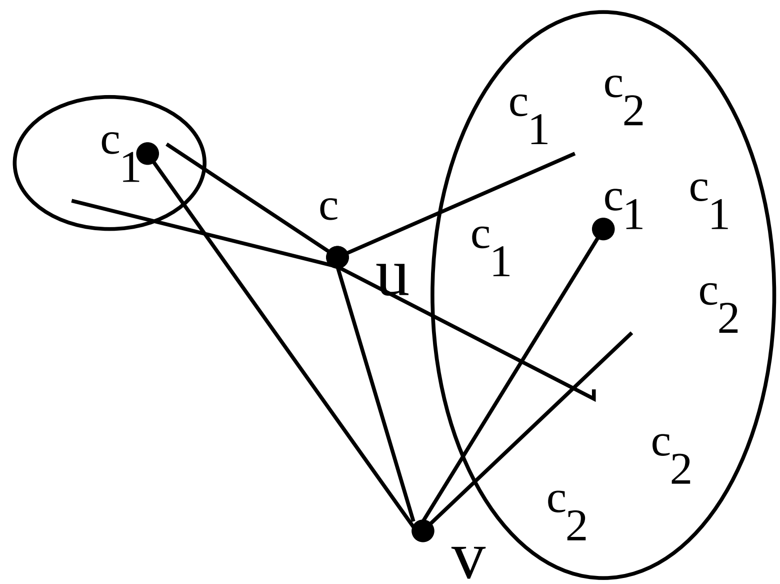
We can therefore assume that G is Δ-regular and 2-connected with $\Delta \geq 3$.

We now consider 2-vertex cutsets. Suppse first that G contains vertices u, v such that $uv \in E$ and u is a cut point of $H = G - v$.



Let $C_1, C_2, \ldots, C_k$ be the components of $H - v$. Each C_i contains at least one neighbour x_i of v, else u is a cutpoint of G.

Take a Δ-colouring of H. Assume first that all neighbours of u have different colours. Interchange colours c_1, c_2 of x_1, x_2 within C_2 only.



Because u does not have colour c_1 or c_2 and C_1 has no neighbours other than u we see that this yields a new proper colouring of H, but now x_1 and x_2 have the same colour c_1.

Thus we can assume that we have a Δ-colouring of H in which 2 neighbours of v have the same colour. This colouring can be extended to v since fewer than Δ colours are being used by neighbours of v.

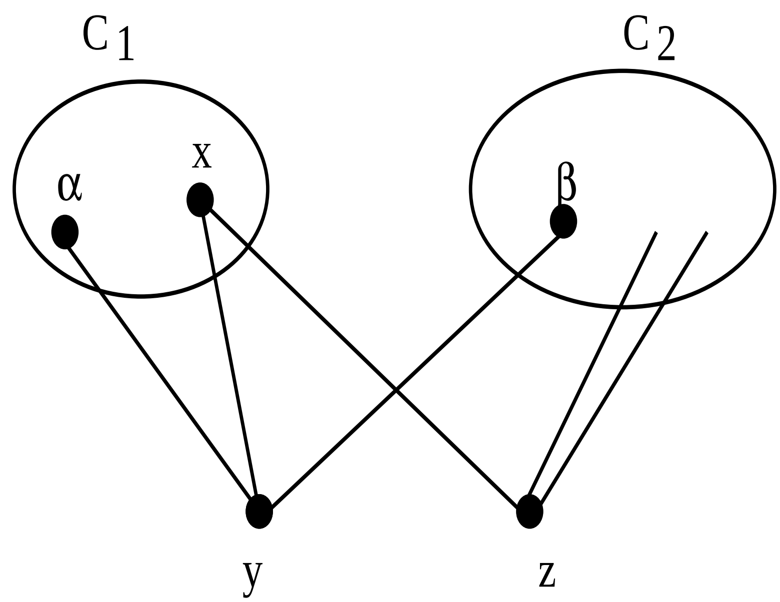
Suppose then that there are no two neighbours which form a 2-vertex cut set. We prove the existence of vertices a, b, c such that

$$ab, ac \in E \text{ and } bc \notin E \text{ and } G - \{b, c\} \text{ is connected.} \tag{2}$$

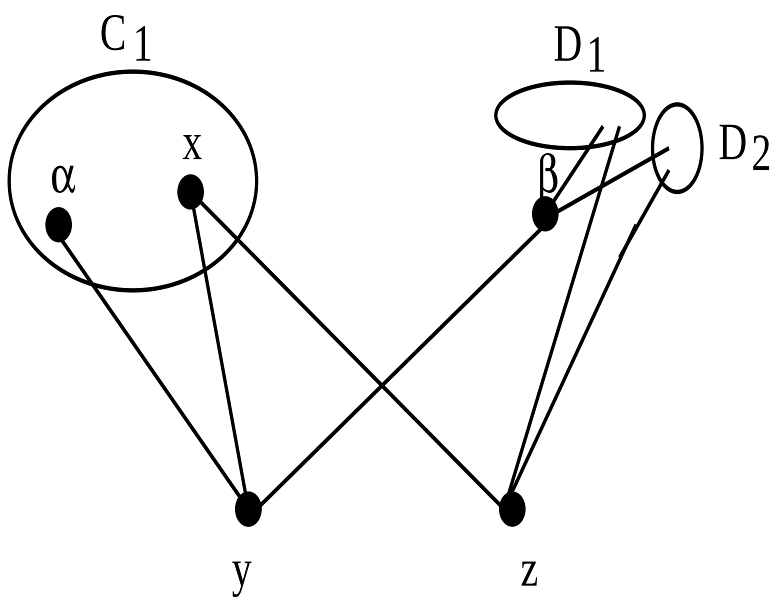
Choose $y \in V$ and let x be at distance 2 from x. y cannot be a neighbour of every other vertex else G is $(\Delta + 1)$-clique. Let x be the middle vertex of a path from x to y of length 2. Then $xy, xz \in E$ and $yz \notin E$.

If $G - \{yz\}$ is connected then let $a, b, c = x, y, z$.

Otherwise let $G-\{yz\}$ have components $C_1, C_2, \ldots, C_k$. y has a neighbour $\alpha \neq x$ in C_1 else x is of degree 2 or is a neighbour of z which is a cutpoint of $G - z$. Similarly, y has a neighbour $\beta \neq x$ in C_2.



We claim that $H = G - \{\alpha, \beta\}$ is connected and so we can take $a, b, c = y, \alpha, \beta$.

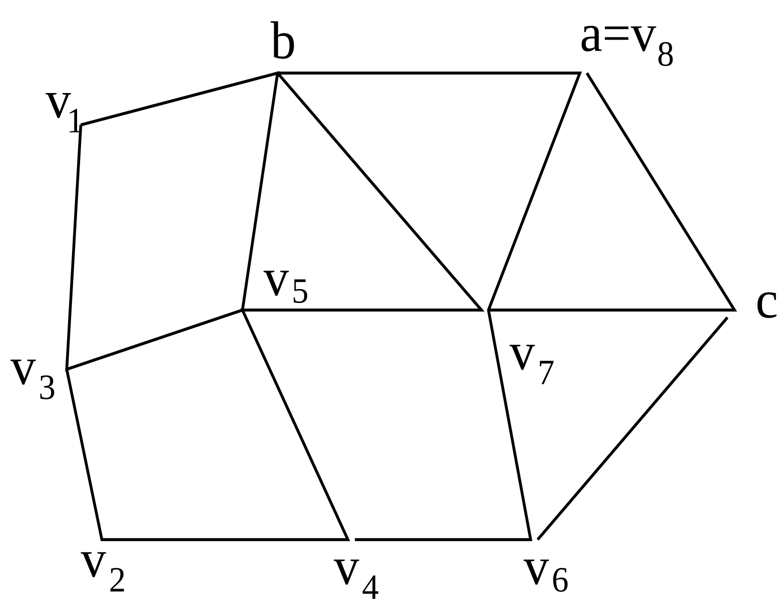


Suppose $C_2 - \beta$ has components $D_1, D_2, \ldots$. Then z is adjacent to D_1 else β is a cutpoint of $G - y$. Similarly, z is adjacent to all components of $C_1 - \alpha$ and $C_2 - \beta$. Now H contains the path x, y, z and every other component $C_3, \ldots, C_k$ is connected to y, z and so H is connected.

Suppose that (2) holds. We run the Greedy colouring algorithm with

$$v_1 = b, v_2 = c, v_3, \dots, v_{n-1}, v_n = a$$

The sequence $v_3, \dots, v_{n-1}, v_n$ is obtained by doing BFS from a in $G - \{b, c\}$.



The important thing is that for $3 \leq i \leq n-1$

$$\exists j > i \text{ such that } v_j \text{ is a neighbour of } v_i. \qquad (3)$$

Greedy uses at most $\triangle$ colours.

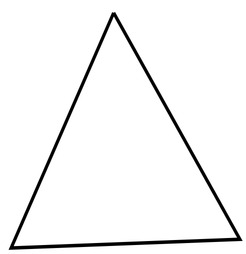
v_1 and v_2 both get colour 1.

For $3 \leq i \leq n-1$, (3) implies that at most $\Delta - 1$ of v_i's neighbours have already been coloured when we come to colour $v-i$.

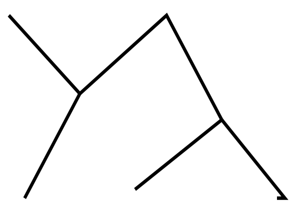
Finally, $v_n = a$ has at least 2 neighbours, b, c using the same colour and so at most $\Delta - 1$ colours have been used so far in a's neighbourhood. $\square$

Chromatic Polynomial

$\pi_k(G)$ is the number of distinct proper k-colourings of G.



$\pi_k = k(k-1)(k-2)$



$\pi_k = k(k-1)^7$

Theorem 4 *Let $e = uv$ be an edge of G. Then*

$$\pi_k(G) = \pi_k(G - e) - \pi_k(G \cdot e).$$

Proof $\pi_k(G)$ = the number of k-colourings of $G - e$ in which u, v have different colours.
$\pi_k(G \cdot e)$ = the number of k-colourings of $G - e$ in which u, v have the same colour. $\square$

Theorem 5 *$\pi_k(G)$ is a polynomial of degree ν in k with integer coefficients, leading term k^ν and constant term zero. The coefficients alternate in sign.*

Proof By induction on $|E|$. If $E = \emptyset$ then $\pi_k(G) = k^\nu$.

Assume true for all graphs with $< m$ edges and let G be a graph with m edges. Then by induction

$$\begin{aligned} \pi_k(G-e) &= k^\nu + \sum_{i=1}^{\nu-1} (-1)^{\nu-i} a_i k^i \\ \pi_k(G \cdot e) &= k^{\nu-1} + \sum_{i=1}^{\nu-2} (-1)^{\nu-1-i} b_i k^i \end{aligned}$$

where $a_1, \ldots, a_{\nu-1}, b_1, \ldots, b_{\nu-2}$ are non-negative integers. Then

$$\pi_k(G) = k^\nu - (a_{\nu-1}+1)k^{\nu-1} + \sum_{i=1}^{\nu-2} (-1)^{\nu-i}(a_i+b_i)k^i.$$

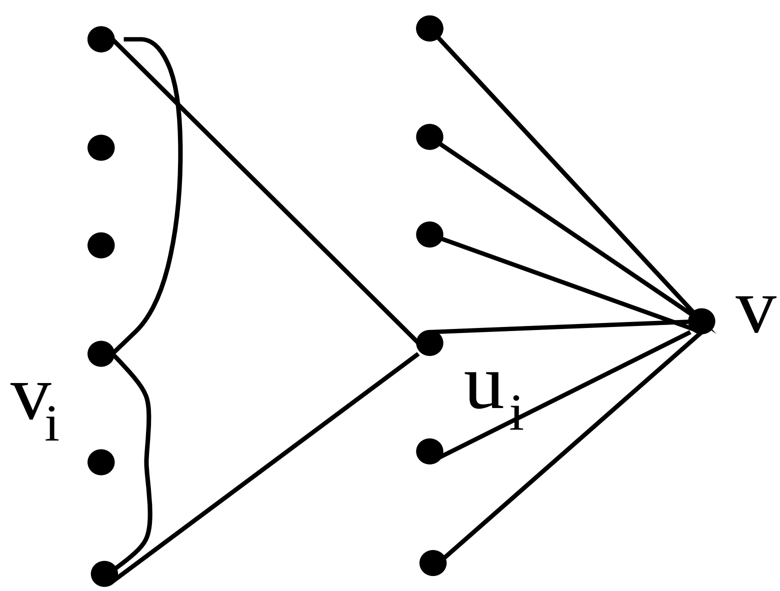
$\square$

Triangle free graphs with high chromatic number

Theorem 6 *For any positive integer k, there exists a triangle-free graph with chromatic number k.*

Proof For $k = 1, 2$ we use K_1, K_2 respectively.

For larger k we use induction on k. Suppose we have a triangle-free graph $G_k = (V_k, E_k)$ of chromatic number k. Let $V_k = \{v_1, v_2, \ldots, v_n\}$. Form G_k as follows:



Add vertices $\{v\} \cup U = \{u_1, u_2, \ldots, u_n\}$ to G_k. Join u_i to v and the neighbours of v_i in G_k, for $1 \leq i \leq n$.

(a) G_{k+1} **has no triangles.**
U is an independent set and so any triangle will have at most one vertex from U. Thus there are no triangles involving v. Finally, if u_i, v_j, v_k is a triangle then v_i, v_j, v_k is a triangle of G_k.

(b) G_{k+1} **does not have a proper k-colouring.**
Suppose there was one c^*. We can assume that $c^*(v) = k$ and then U is coloured from $\{1, 2, \ldots, k-1\}$. But now we can define a proper $(k-1)$-colouring c of G_k by

$$c(v_i) = \begin{cases} c^*(v_i) & \text{if } c^*(v_i) \neq k \\ c^*(u_i) & \text{if } c^*(v_i) = k \end{cases}$$

This is a proper colouring of G_k since if $v_i v_j$ is an edge of G_k with $c(v_i) = c(v_j)$ then exactly one of $c(v_i) \neq c^*(v_i)$ or $c(v_j) \neq c^*(v_j)$ holds. Assume the former. Then $c^*(v_i) = k$ and $c(v_i) = c^*(u_i) \neq c^*(v_j) = c(v_j)$. Thus G_{k+1} is k-colourable implies G_k is $(k-1)$-colourable, which it isn't.

(c) G_{k+1} **has a proper** $(k+1)$**-colouring.**

Let c be a proper k-colouring of G_k. Extend this to U by putting $c(u_i) = c(v_i)$ and then let $c(v) = k + 1$. Note that u_i and v_i have the same colour and the same neighbours in V_k and so the colouring remains proper.