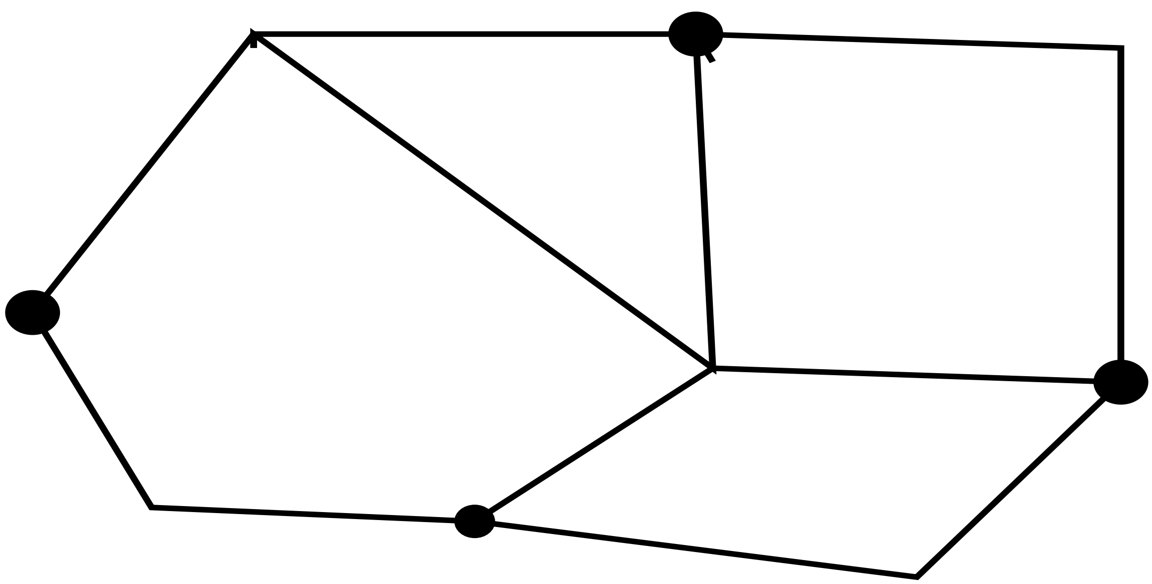
Independent sets and cliques

$S \subseteq V$ is *independent* if no edge of G has both of its endpoints in S.



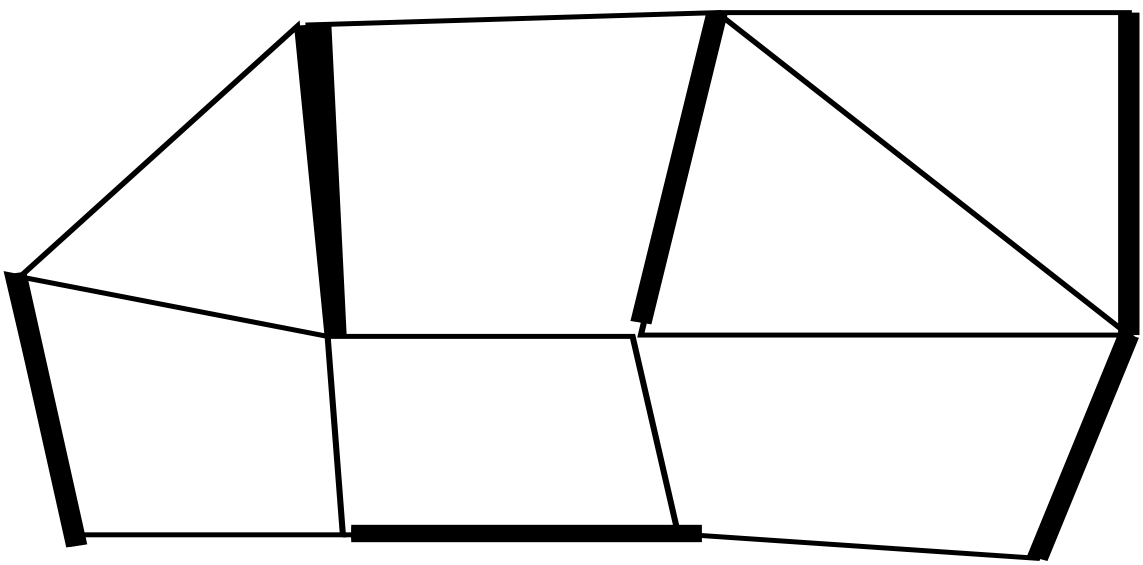
$\alpha(G)$=maximum size of an independent set of G.

Lemma 1 *S is independent iff $V \setminus S$ is a cover.*

Corollary 1

$$\alpha(G) + \beta(G) = \nu.$$

$L \subseteq E$ is an *edge covering* if every $v \in V$ is contained in an edge of L.



$\beta'(G)$=minimum size of an edge cover
$\alpha'(G)$=maximum size of a matching.

Theorem 1 *If there are no isolated vertices then*

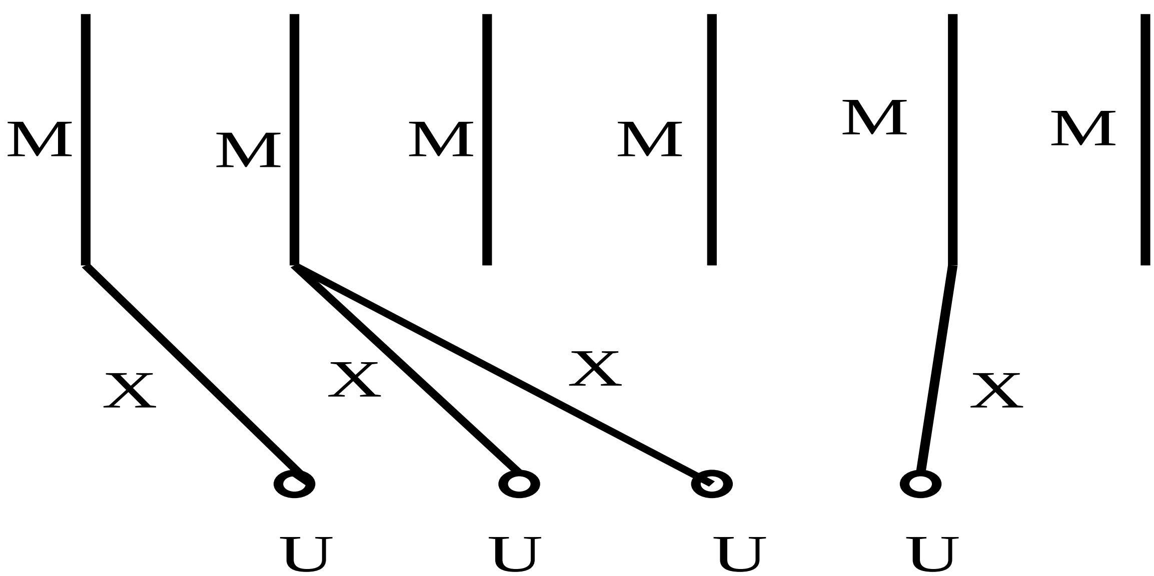
$$\alpha' + \beta' = \nu.$$

Proof
(a) $\alpha' + \beta' \leq \nu$.

Let M be a maximum matching of G.
Let U be the set of vertices unsaturated by M.

Cover U with edges X, $|X| = |U|$.

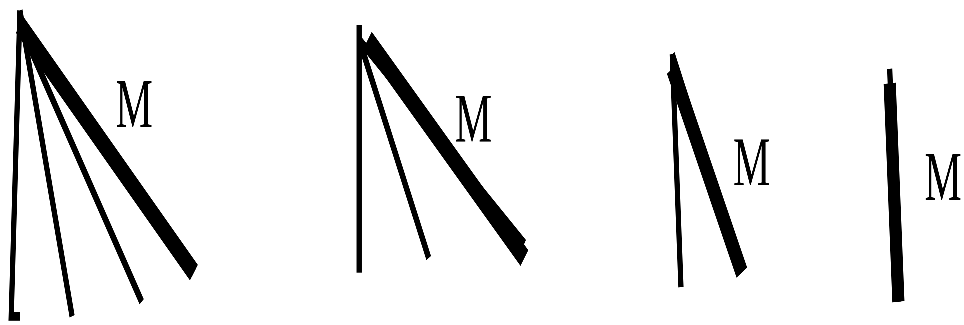
$M \cup X$ is a cover.

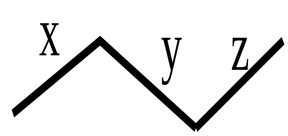


$$\begin{aligned}\beta' &\leq |M| + |X| \\ &= \alpha' + (\nu - 2\alpha') \\ &= \nu - \alpha'.\end{aligned}$$

(b) $\alpha' + \beta' \geq \nu$.

Let L be a minimum edge cover of G.
$G[L]$ is a collection of disjoint stars $S_1, S_2, \ldots, S_k$.



[If G[L] contained 

then L-y is a smaller cover.]

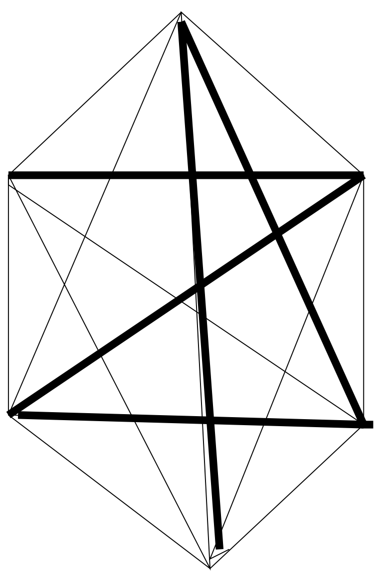
Choose matching M, one edge from each S_i.

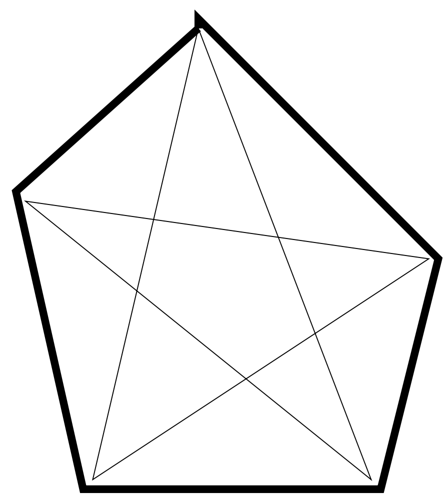
$$\begin{aligned} \beta' = |L| &= \nu - k \\ &= \nu - |M| \\ &\geq \nu - \alpha' \end{aligned}$$

$\square$

Ramsey's Theorem

Suppose we 2-colour the edges K_6 of Red and Blue. There *must* be either a Red triangle or a Blue triangle.





This is not true for K_5.

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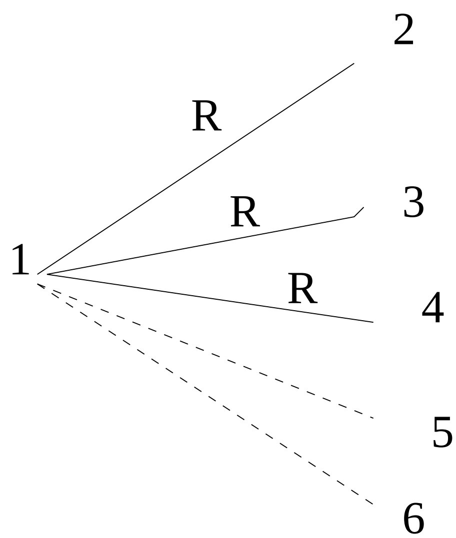


There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter implies (1,2,3) is a Red triangle.

Ramsey's Theorem

For all positive integers k, ℓ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of K_N are coloured Red or Blue then then either there is a "Red k-clique" or there is a "Blue ℓ-clique.

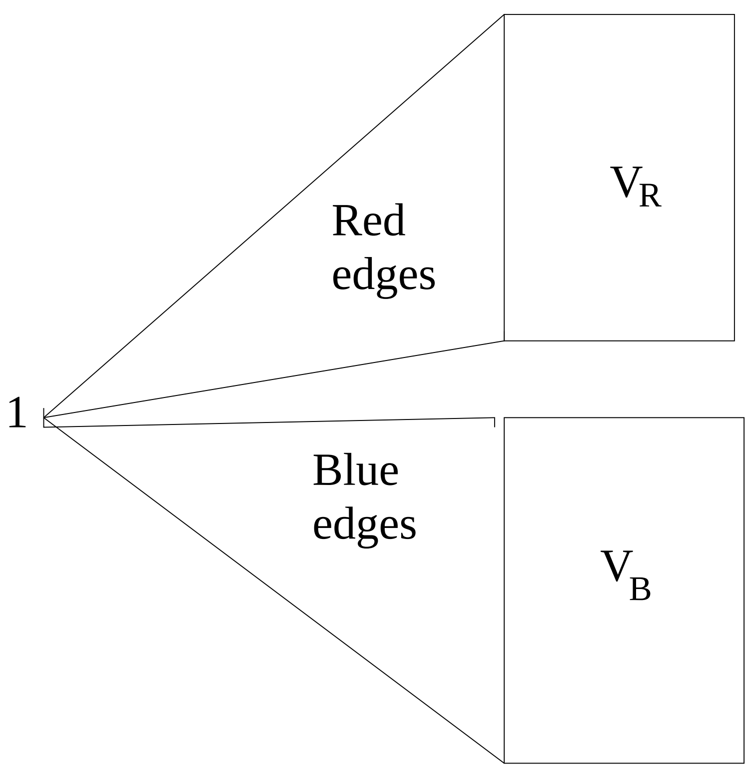
A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

$$\begin{aligned} R(1,k) &= R(k,1) &= 1 \\ R(2,k) &= R(k,2) &= k \end{aligned}$$

Theorem 2

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell).$$

Proof Let $N = R(k,\ell-1) + R(k-1,\ell)$.



$V_R = \{(x :\ (1,x)$ is coloured Red$\}$ and $V_B = \{(x :$ $(1,x)$ is coloured Blue$\}$.

$$|V_R| \geq R(k-1,\ell) \text{ or } |V_B| \geq R(k,\ell-1).$$

Since

$$\begin{aligned} |V_R| + |V_B| &= N-1 \\ &= R(k,\ell-1) + R(k-1,\ell) - 1. \end{aligned}$$

Suppose for example that $|V_R| \geq R(k-1,\ell)$. Then either V_R contains a Blue ℓ-clique – done, or it contains a Red $k-1$-clique K. But then $K \cup \{1\}$ is a Red k-clique.

Similarly, if $|V_B| \geq R(k,\ell-1)$ then either V_B contains a Red k-clique – done, or it contains a Blue $\ell-1$-clique L and then $L \cup \{1\}$ is a Blue ℓ-clique. □

Theorem 3

$$R(k,\ell) \le \binom{k+\ell-2}{k-1}.$$

Proof Induction on $k+\ell$. True for $k+\ell \le 5$ say. Then

$$\begin{aligned} R(k,\ell) &\le R(k,\ell-1) + R(k-1,\ell) \\ &\le \binom{k+\ell-3}{k-1} + \binom{k+\ell-3}{k-2} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

□

So, for example,

$$\begin{aligned} R(k,k) &\le \binom{2k-2}{k-1} \\ &\le 4^k \end{aligned}$$

Theorem 4

$$R(k,k) > 2^{k/2}$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.

We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: $\{$There is a Red k-clique$\}$ and
$\mathcal{E}_B$ be the event: $\{$There is a Blue k-clique$\}$.

We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.
Let $\mathcal{E}_{R,j}$ be the event: $\{C_j$ is Red$\}$.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) \\
&= 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \\
&\leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
$$

□

More than two colours

$n \geq R(k_1, k_2, \ldots, k_m)$ implies that if the edges of K_n are coloured with $\{1, 2, \ldots, m\}$ then $\exists i : K_n$ contains a k_i-clique all of whose edges have colour i. These numbers exist and satisfy

Theorem 5 *(a)*

$$R(k_1, k_2, \ldots, k_m) \leq R(k_1 - 1, k_2, \ldots, k_m) + R(k_1, k_2 - 1, \ldots, k_m) + \cdots + R(k_1, k_2, \ldots, k_m - 1) - (m - 2).$$

(b)

$$R(k_1, k_2, \ldots, k_m) \leq \frac{(k_1 + k_2 + \cdots + k_m - m)!}{(k_1 - 1)!(k_2 - 1)! \cdots (k_m - 1)!}.$$

□

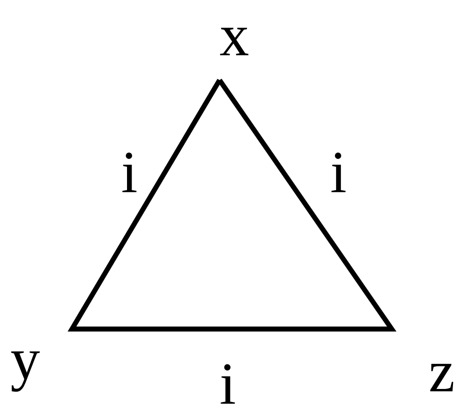
Schur's Theorem

Theorem 6 *For any* $k \geq 1$ *there exists an integer* f_k *such that for any partition* $S_1, S_2, \ldots, S_k$ *of* $\{1, 2, \ldots, f_k\}$ *there exists an* i *and* $a, b, c \in S_i$ *such that* $a + b = c$*.*

Proof Let $f = f_k = R(3, 3, \ldots, 3)$. Edge colour K_f by

$$xy \text{ gets colour } i \text{ iff } |x - y| \in S_i.$$

There exists i such that a triangle is coloured i.



$x<y<z$

$$\begin{aligned} a &= y - x \in S_i \\ b &= z - y \in S_i \\ c &= z - x \in S_i \\ a + b &= c \end{aligned}$$

Turan's Theorem

A graph is $K_m - free$ if it contains no clique of size m (or more).

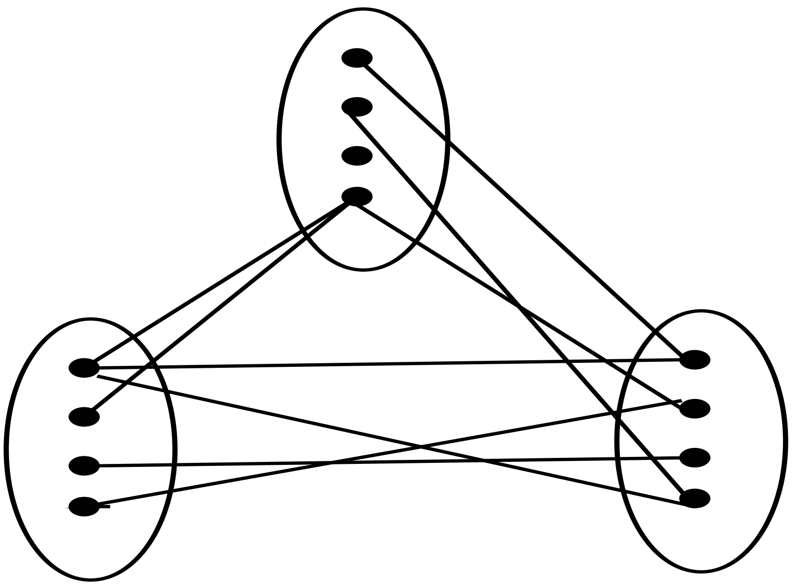
How many edges can a there be in a $K_m - free$ graph?

$m = 3$ – triangle free.

$K_{\lfloor \nu/2 \rfloor, \lceil \nu/2 \rceil}$ has no triangles and no triangle free graph with ν vertices has more edges.

t-partite graphs

G is t-partite if $V = V_1 \cup V_2 \cup \cdots \cup V_t$ is a partition where $V_1, V_2, \ldots, V_t$ are independent sets.



3-partite

A t-partite graph is K_{t+1}-free — pigeon hole principle.

$K_{m_1,m_2,\ldots,m_t}$ is a *complete* t-partite graph.
$|V_i| = m_i$ for $1 \leq i \leq t$.
Every vertex in V_i is connected to every vertex in V_j by an edge, $1 \leq i < j \leq t$.

Therefore

$$\epsilon(K_{m_1,m_2,\ldots,m_t}) = \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} m_i m_j.$$

Which ν vertex t-partite graph has most edges?

Suppose $\nu = kt + \ell$ where $0 \leq \ell < t$.

$$T_{t,\nu} = K_{k,k,\ldots,k+1}$$

($t-\ell$ k's and ℓ $k+1$'s in the sequence $k, k, \ldots, k+1$.)

Lemma 2 *If* $m_1 + m_2 + \cdots + m_t = \nu$ *then*

$$\epsilon(K_{m_1,m_2,\ldots,m_t}) < \epsilon(T_{t,\nu})$$

unless $K_{m_1,m_2,\ldots,m_t} \cong T_{t,\nu}$.

Proof Suppose that $m_2 \geq m_1 + 2$. Then

$$\begin{aligned}\epsilon(K_{m_1+1,m_2-1,\ldots,m_t}) &= \epsilon(K_{m_1,m_2,\ldots,m_t}) + \\ &\qquad + m_2 - m_1 - 1 \\ &> \epsilon(K_{m_1,m_2,\ldots,m_t}).\end{aligned}$$

So if the block sizes are not as even as possible, the number of edges is not maximum. □

$G_1 = (V, E_1)$ *degree majorises* $G_2 = (V, E_2)$ if

$$d_{G_1}(v) \geq d_{G_2}(v) \qquad \text{for all } v \in V.$$

We write $G_1 \geq_{dm} G_2$.

Theorem 7 *If G is simple and K_{m+1} free then there exists a complete m-partite graph H such that*

(a) $H \geq_{dm} G$.

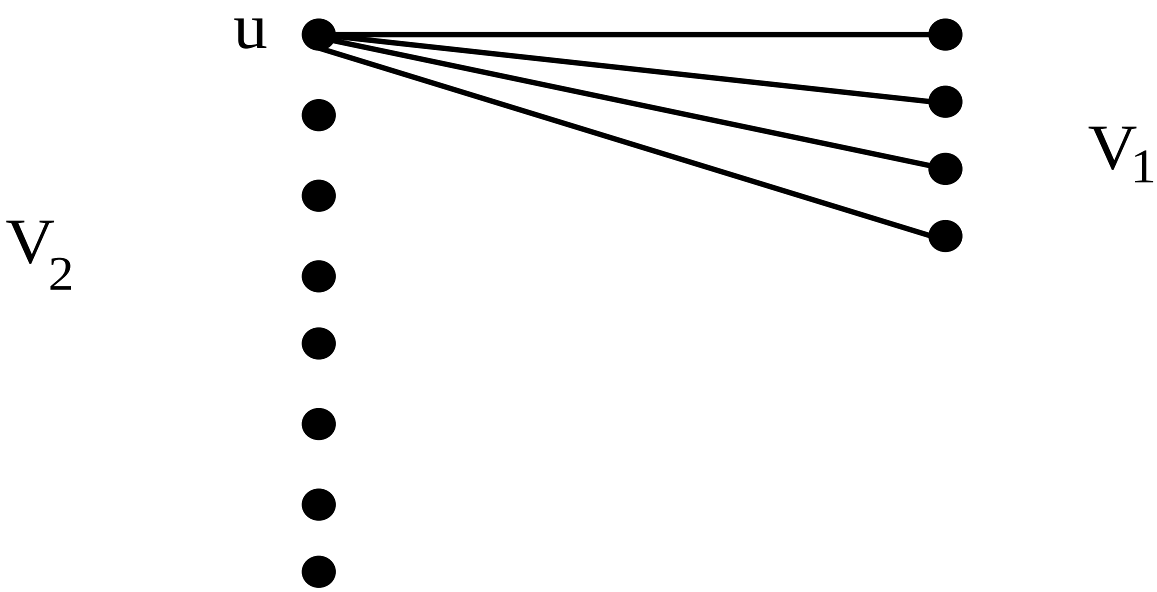
(b) $\epsilon(G) = \epsilon(H)$ *implies that* $G \cong H$.

Proof By induction on m.

True for $m = 1$ as K_2-free means $E = \emptyset$.

Assume the result for $m' < m$ and let G be K_{m+1}-free.

Let $d_G(u) = \Delta(G)$, $V_1 = N(u)$, $|V_1| = \Delta$ and $V_2 = V \setminus V_1$.

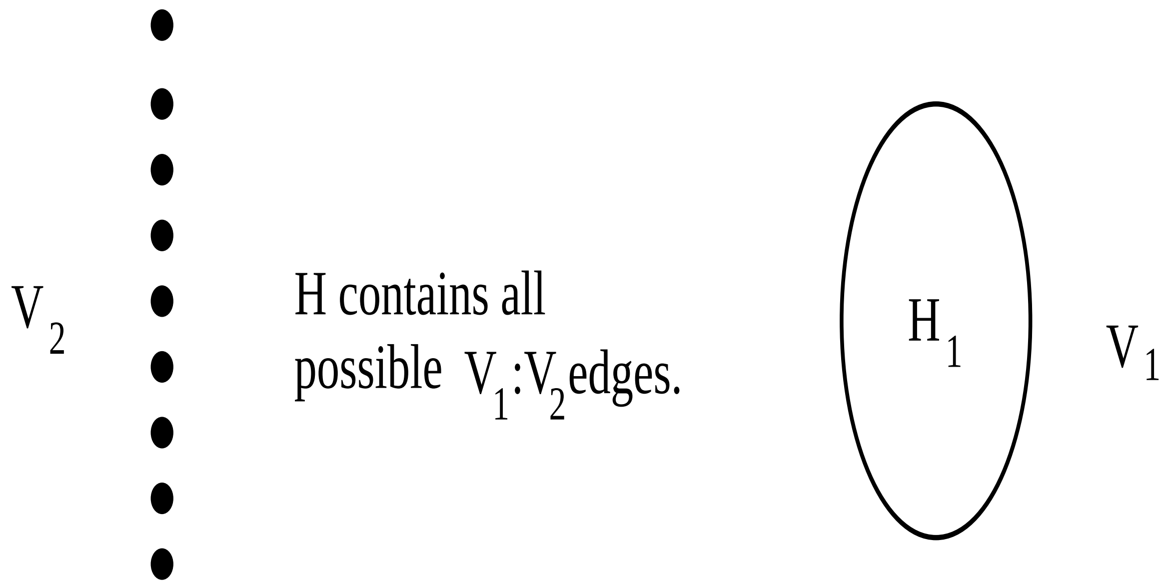


$G_1 = G[V_1]$ is K_m-free.

There is a complete $(m-1)$-partite graph H_1 such that $H_1 \geq_{dm} G_1$ — induction.

Let

$$H = V_2 \wedge G_1.$$



We claim that

$$H \geq_{dm} G.$$

$$\begin{aligned} v \in V_2 \text{ implies } d_G(v) &\leq \Delta = d_H(v) \\ v \in V_1 \text{ implies } d_G(v) &\leq |V_2| + d_{G_1}(v) \\ &\leq |V_2| + d_{H_1}(v) \\ &= d_H(v) \end{aligned}$$

(b) Now suppose that $\epsilon(G) = \epsilon(H)$. This implies that $d_G(v) = d_H(v)$ for all $v \in V$.

Let t be the number of edges contained in V_2. We claim that $t = 0$.

$$\begin{aligned} \Delta|V_2| &= 2t + |V_2 : V_1| \\ \epsilon(G) &= t + |V_2 : V_1| + \epsilon(G_1) \\ \epsilon(H) &= \Delta|V_2| + \epsilon(H_1). \end{aligned}$$

So $0 \leq t = \epsilon(G_1) - \epsilon(H_1) \leq 0$. Thus $\epsilon(G_1) = \epsilon(G_2)$ and V_2 is an independent set in G. We can now use induction to argue that $G_1 \cong H_1$ and then $G \cong H$. □

Theorem 8 *If G is simple and K_{m+1}-free then*

(a) $\epsilon(G) \leq \epsilon(T_{m,\nu})$.

(b) $\epsilon(G) = \epsilon(T_{m,\nu})$ *imlpies that* $G \cong T_{m,\nu}$.

Proof (a) follows from Lemma 2 and Theorem 7a. For (b) we observe that the graph H of Theorem 7 satisfies

$$\begin{aligned} \epsilon(G) &= \epsilon(H) = \epsilon(T_{m,\nu}) \\ G &\cong H \end{aligned}$$

But then $\epsilon(H) = \epsilon(T_{m,\nu})$ and Lemma 2 implies that $H \cong T_{m,\nu}$. □

Geometry Problem

Theorem 9 *Let* $X_1, X_2, \ldots, X_n$ *be points in the plane such that for* $1 \leq i < j \leq n$

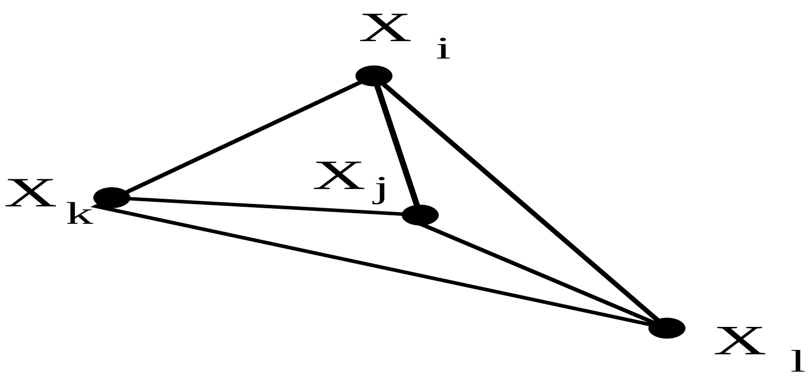
$$|X_i - X_j| \leq 1.$$

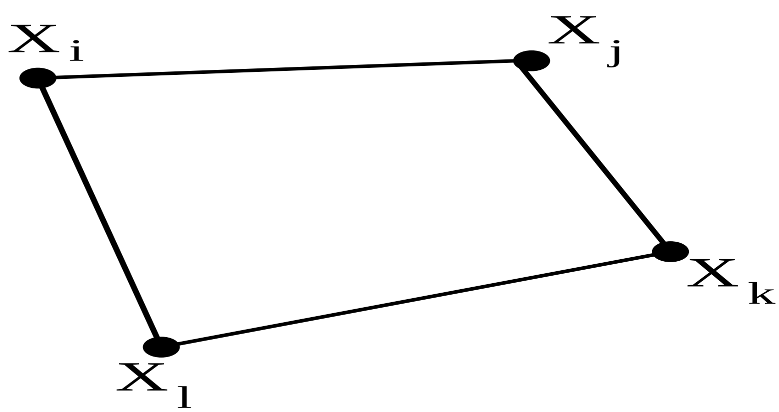
Then

$$|\{(i,j) : i < j \text{ and } |X_i - X_j| > 1/\sqrt{2}\}| \leq \lfloor n^2/3 \rfloor.$$

Proof Define graph G with $V = \{1, 2, \ldots, n\}$ and $E = \{(i,j) : |X_i - X_j| > 1/\sqrt{2}\}$. We claim that G has no K_4 and so

$$|E| \leq \epsilon(T_{3,n}) = \lfloor n^2/3 \rfloor.$$

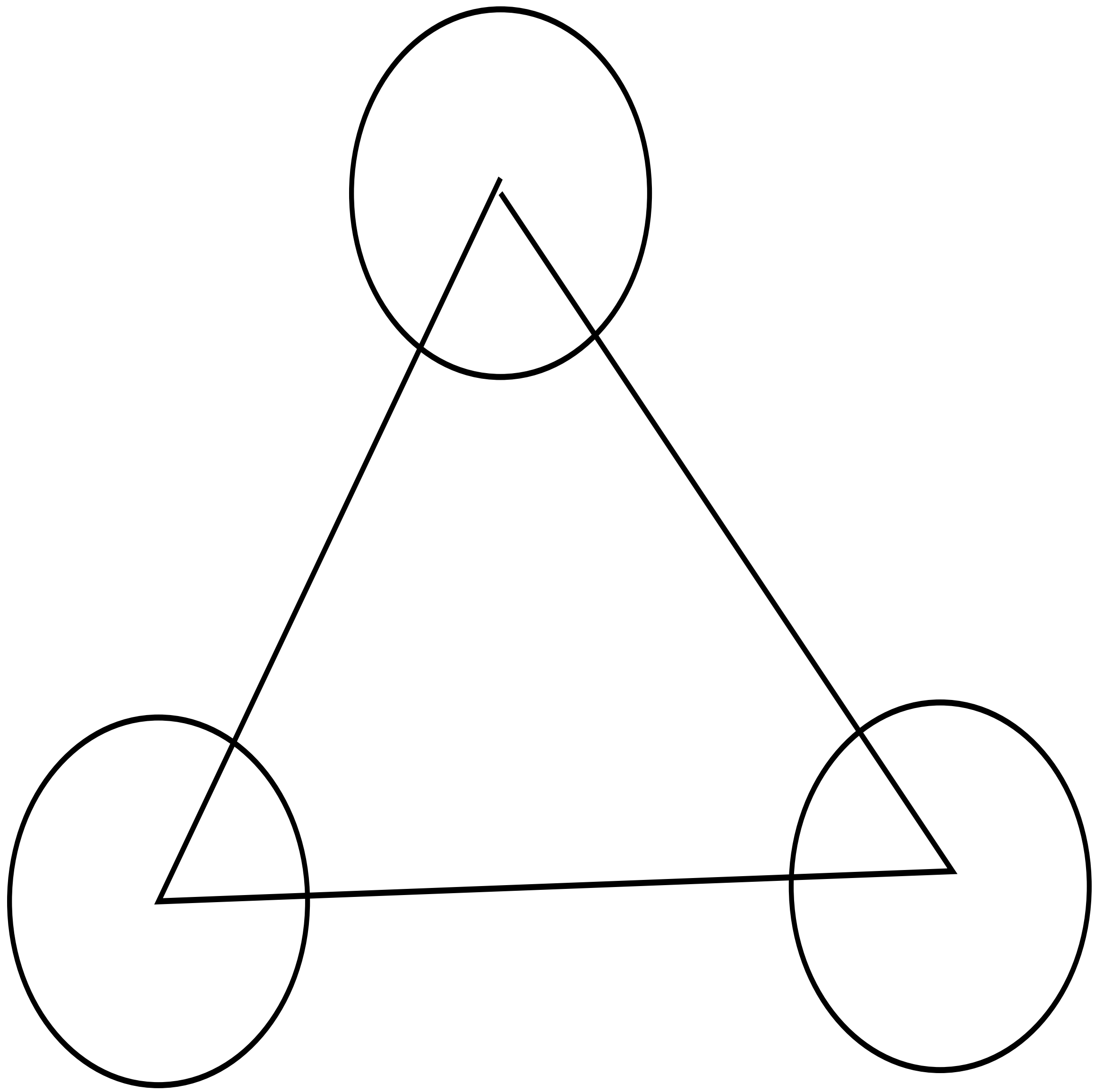




There exist i, j, k such that $\angle X_1 X_j X_k \geq \pi/2$. Then

$$1 \geq |X_i X_k|^2 \geq |X_i X_j|^2 + |X_j X_k|^2.$$

$\square$



The circles are of radius r and the sides of the triangle are $1 - 2r$ where $0 < r < (1 - 1/\sqrt{2})/4$. The n points are split as evenly as possible within each circle.

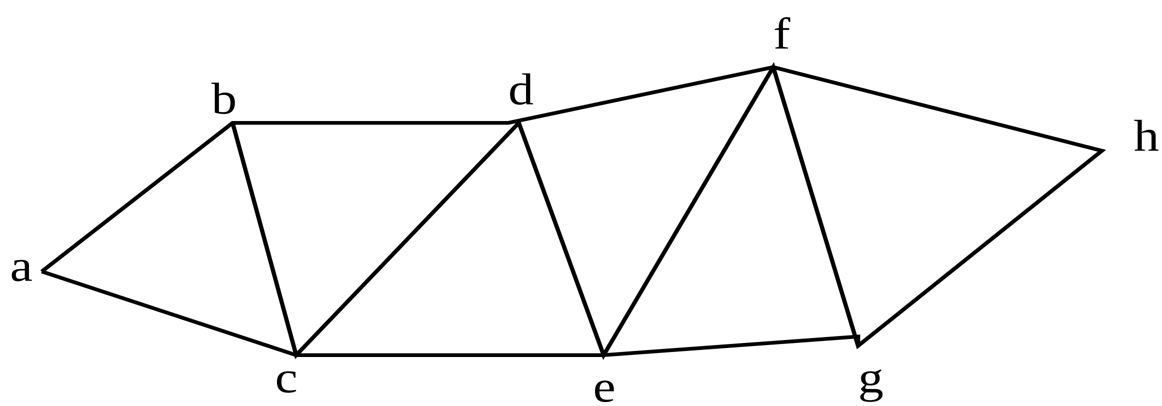
Theorem 10 *If* $\bar{d} = 2\epsilon/\nu$ *= the average degree of simple graph* G *then*

$$\alpha(G) \geq \frac{\nu}{\bar{d}+1}.$$

Proof Let $\pi(1), \pi(2), \ldots, \pi(\nu)$ be an arbitrary permutation of V. Let $N(v)$ denote the set of neighbours of vertex v and let

$$I(\pi) = \{v : \pi(w) > \pi(v) \text{ for all } w \in N(v)\}.$$

Claim 1 I *is an independent set.*



	a	b	c	d	e	f	g	h	I
π_1	c	b	f	h	a	g	e	d	$\{c, f\}$
π_2	g	f	h	d	e	a	b	c	$\{g, d, a\}$

Proof of Claim 1
Suppose $w_1, w_2 \in I(\pi)$ and $w_1 w_2 \in E$. Suppose $\pi(w_1) < \pi(w_2)$. Then $w_2 \notin I(\pi)$ — contradiction.
$\square$

Now let π be a random permutation.

Claim 2

$$\mathbf{E}(|I|) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

Proof of Claim 2
Let

$$\delta(v) = \begin{cases} 1 & v \in I \\ 0 & v \notin I \end{cases}$$

Thus

$$\begin{aligned} |I| &= \sum_{v \in V} \delta(v) \\ \mathbf{E}(|I|) &= \sum_{v \in V} \mathbf{E}(\delta(v)) \\ &= \sum_{v \in V} \mathbf{Pr}(\delta(v) = 1). \end{aligned}$$

Now $\delta(v) = 1$ if v comes before all of its neighbours in the order π. Thus

$$\mathbf{Pr}(\delta(v) = 1) \geq \frac{1}{d(v) + 1}$$

and the claim follows. □

Thus there exists a π such that

$$|I(\pi)| \geq \sum_{v \in V} \frac{1}{d(v) + 1}$$

and so

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1}.$$

We finish the proof of Theorem 10 by showing that

$$\sum_{v \in V} \frac{1}{d(v) + 1} \geq \frac{\nu}{\bar{d} + 1}.$$

This follows from the following claim by putting $x_v = d(v) + 1$ for $v \in V$.

Claim 3 *If* $x_1, x_2, \ldots x_k > 0$ *then*

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \geq \frac{k^2}{x_1 + x_2 + \cdots + x_k}. \quad (1)$$

Proof of Claim 3

Multiplying (1) by $x_1 + x_2 + \cdots + x_k$ and subtracting k from both sides we see that (1) is equivalent to

$$\sum_{1 \le i < j \le k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right) \ge k(k-1). \tag{2}$$

But for all $x, y > 0$

$$\frac{x}{y} + \frac{y}{x} \ge 2$$

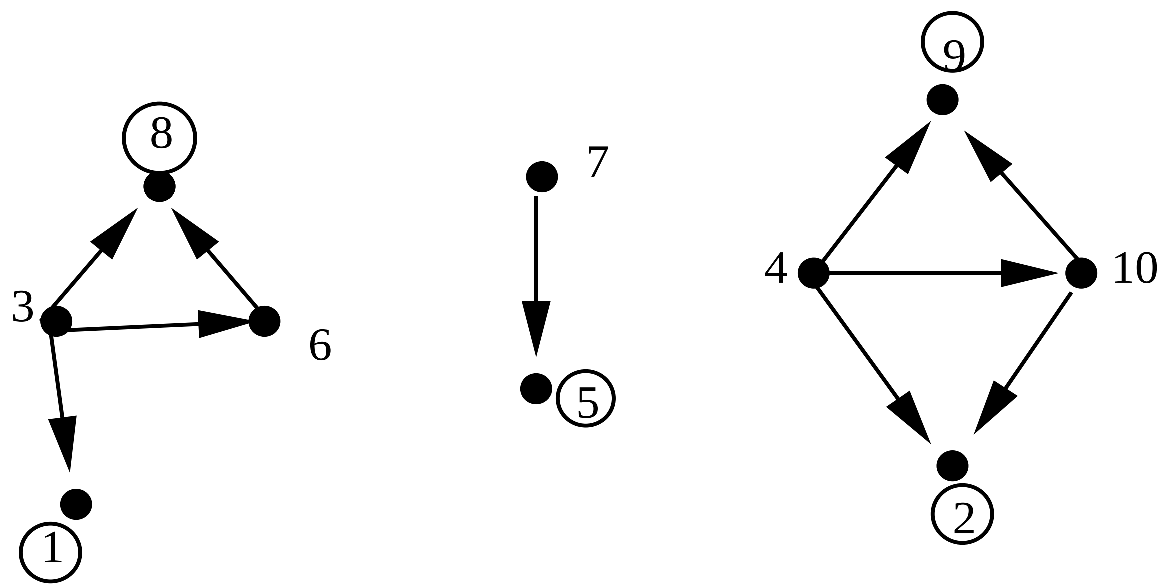
and (2) follows. □

Parallel searching for the maximum – Valiant

We have n processors and n numbers $x_1, x_2, \dots, x_n$.
In each round we choose n pairs i, j and compare the values of x_i, x_j.
The set of pairs chosen in a round can depend on the results of previous comparisons.

Aim: find i^* such that $x_{i^*} = \max_i x_i$.

Claim 4 *For any algorithm there exists an input which requires at least* $\frac{1}{2} \log_2 \log_2 n$ *rounds.*



Suppose that the first round of comparisons involves comparing x_i, x_j for edge ij of the above graph and that the arrows point to the larger of the two values. Consider the independent set $\{1, 2, 5, 8, 9, \}$. These are the indices of the 5 largest elements, but their relative order can be arbitrary since there is no implied relation between their values.

Let $C(a, b)$ be the maximum number of rounds needed for a processors to compute the maximum of b values in this way.

Lemma 3

$$C(a, b) \geq 1 + C\left(a, \left\lceil \frac{b^2}{2a+b} \right\rceil\right).$$

Proof The set of b comparisons defines a b-edge graph G on a vertices where comparison of x_i, x_j produces an edge ij of G. Theorem 10 implies that

$$\alpha(G) \geq \left\lceil \frac{b}{\frac{2a}{b} + 1} \right\rceil = \left\lceil \frac{b^2}{2a+b} \right\rceil.$$

For any independent set I it is always possible to define values for $x_1, x_2, \ldots, x_a$ such I is the index set of the $|I|$ largest values and so that the comparisons do not yield any information about the ordering of the elements $x_i, i \in I$.

Thus after one round one has the problem of finding the maximum among $\alpha(G)$ elements. $\square$

Now define the sequence $c_0, c_1, \ldots$ by $c_0 = n$ and

$$c_{i+1} = \left\lceil \frac{c_i^2}{2n + c_i} \right\rceil .$$

It follows from Lemma 3 that

$$c_k \geq 2 \text{ implies } C(n,n) \geq k+1.$$

Claim 4 now follows from

Claim 5

$$c_i \geq \frac{n}{3^{2^i - 1}}.$$

By induction on i. Trivial for $i = 0$. Then

$$\begin{aligned} c_{i+1} &\geq \frac{n^2}{3^{2^{i+1}-2}} \times \frac{1}{2n + \frac{n}{3^{2^i-1}}} \\ &= \frac{n}{3^{2^{i+1}-1}} \times \frac{3}{2 + \frac{1}{3^{2^i-1}}} \\ &\geq \frac{n}{3^{2^{i+1}-1}}. \end{aligned}$$

□