Edge Colourings

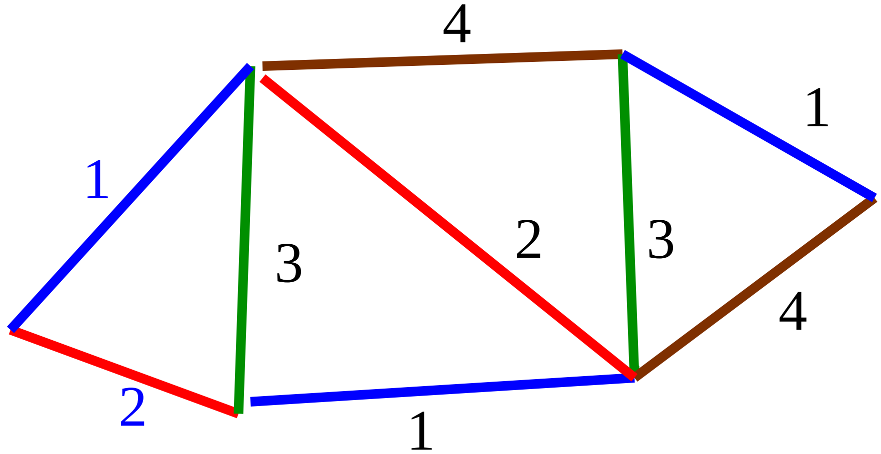
We assume in this chapter that G has no loops.

A $k-edge\ colouring$ of G is a mapping

$$c:\ E \rightarrow \{1,2,\dots,k\}.$$

$c(e)$ is the colour of edge e.

$M_i = \{e \in E:\ c(e) = i\}$ is the set of edges with colour i.



c is *proper* if $M_1, M_2, \dots, M_k$ are matchings i.e. edges e, f sharing a common vertex have $c(e) \neq c(f)$.

G is $k-edge\ colourable$ if it has a proper k-edge colouring.

$$\chi'(G) = \min\{k :\ G \text{ is } k\text{-edge colourable}\}.$$

Lemma 1

$$\chi'(G) \geq \Delta(G).$$

Proof If $d(v) = \Delta$ then every edge incident with v must have a distinct colour in a proper edge colouring. □

Lemma 2 *If G' is a subgraph of G then*

$$\chi'(G) \geq \chi'(G').$$

Proof A proper colouring of G induces a proper colouring of G'. □

Bipartite Graphs

Theorem 1 *If G is a k-regular bipartite graph then $\chi'(G) = k$.*

Proof $\chi'(G) \geq k$ by Lemma 1. We prove by induction on k that G has a proper k-colouring.

$k = 1$: G is a matching covering all vertices and so is 1-edge colourable.

Assume that $\chi'(H) = \ell$ for all ℓ-regular bipartite graphs with $\ell < k$.

G contains a perfect matching M.
$G - M$ is $(k-1)$-regular and so, by the inductive hypothesis, has a proper $(k-1)$-edge colouring c'. Define a proper k-edge colouring c of G by

$$c(e) = \begin{cases} c'(e) & e \notin M \\ k & e \in M \end{cases}$$

□

Corollary 1 *If G is bipartite then $\chi'(G) = \Delta$.*

Proof We add edges to G to produce a Δ-regular bipartite graph G'.
(Repeatedly join pairs of vertices of degree $<$ Δ until the graph is Δ-regular.)

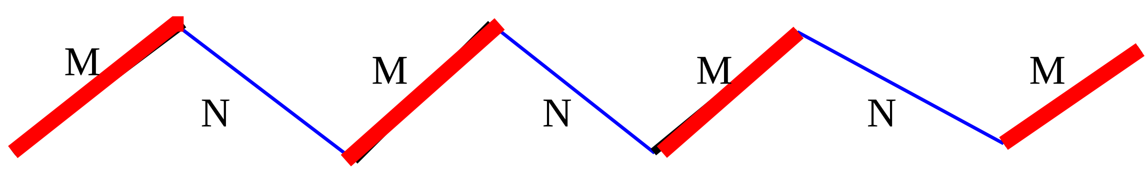
Then

$$\Delta \leq \chi'(G) \leq \chi'(G') = \Delta.$$

□

Lemma 3 *Let M, N be disjoint matchings of G with $|M| > |N|$. Then there exist disjoint matchings M', N' such that (i) $M' \cup N' = M \cup N$ and (ii) $|M'| = |M| - 1$, $|N'| = |N| + 1$.*

Proof $G[M \cup N]$ contains at least one alternating path P which starts and ends with M-edges.



Let $M' = M \Delta P$ and $N' = N \Delta P$ i.e. remove the M-edges of P from M and replace them by the N-edges of P to obtain M'. Remove the N-edges of P from N and replace them by the M-edges of P to obtain N'. □

Theorem 2 *If G is a bipartite graph and $p \geq \Delta$ then there exists a p-edge colouring $M_1 \cup M_2 \cup \cdots \cup M_p$ such that*

$$\lfloor |E|/p \rfloor \leq |M_i| \leq \lceil |E|/p \rceil \qquad 1 \leq i \leq p. \tag{1}$$

Proof Start with an arbitrary proper p-edge colouring of E (some colour classes may be empty.) If there exist a pair of matchings M_i, M_j which differ in size by 2 or more then use Lemma 3 to reduce the larger and increase the smaller. This yields a new proper edge colouring.

Repeat until (1) holds. □

School Timetabling

m teachers $A_1, A_2, \ldots, A_m$.
n classes $B_1, B_2, \ldots, B_n$.
A_i teaches class B_j $p_{i,j}$ times.
r rooms available.

Let

$$\begin{aligned} \Delta &= \max\left\{\max_{i=1}^{m}\sum_{j=1}^{n} p_{i,j}, \max_{j=1}^{n}\sum_{i=1}^{m} p_{i,j}\right\} \\ &= \text{maximum class/teacher load} \end{aligned}$$

$$\begin{aligned} \ell &= \sum_{i=1}^{m}\sum_{j=1}^{n} p_{i,j} \\ &= \text{total number of classes} \end{aligned}$$

Clearly we need at least

$$p = \max\{\Delta, \lceil \ell/r \rceil\}$$

periods.

Theorem 3 *There is a feasible p period timetable.*

Proof Define the bipartite graph G with $A = \{A_1, A_2, \dots, A_m\}$, $B = \{B_1, B_2, \dots, B_n\}$ and $p_{i,j}$ edges joining A_i and B_j.

G has maximum degree Δ.

By Theorem 2 G has a p-edge colouring $M_1, M_2, \dots, M_p$ with

$$|M_i| \leq \lceil \ell/p \rceil \leq \lceil \ell/\lceil \ell/r \rceil \rceil \leq r.$$

Each M_i represents the teaching of a particular period. □

Vizing's Theorem

If G is an odd cycle then $\chi'(G) = 3 > \Delta(G) = 2$.

Theorem 4 *If G is simple then*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Proof We need to prove the existence of a proper $(\Delta + 1)$-edge colouring. We prove this by induction on $|V|$. It is clearly true for $|V| = 1$.

Assume inductively that the theorem is true for all simple graphs with fewer than n vertices and suppose that $|V| = n$.

For $v \in V$ let $G' = G - v$.

$\chi'(G') \leq \Delta(G')+1 \leq \Delta(G)+1$ induction.

Thus there is a $k = \Delta + 1$ proper edge colouring of the edges of G'.

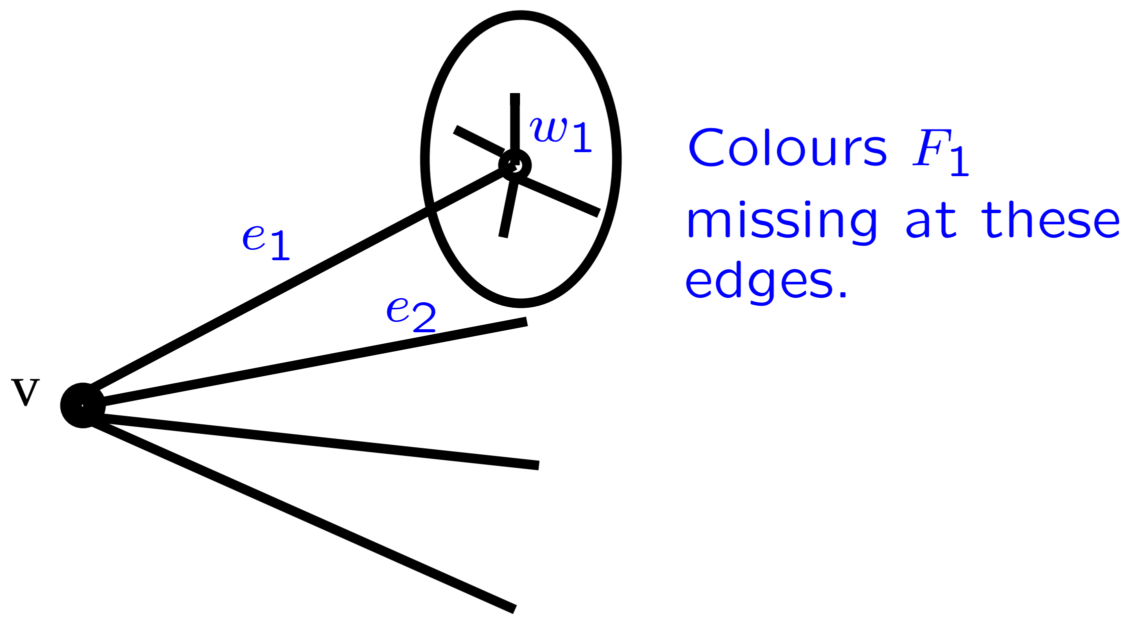
Viziing's theorem follows from

Lemma 4 *Let G be a simple graph, $v \in V$ and $e_1, e_2, \ldots, e_r \in E$ be incident with v where $e_i = vw_i, 1 \leq i \leq r$ and $w_0 = v$.*

Suppose $k > \Delta(G)$ and $G^* = G - \{e_1, e_2, \ldots, e_r\}$ is k-edge colourable with the following property: F_i is the set of colours not used on the edges incident with w_i for $0 \leq i \leq r$.

$|F_i \cap F_0| \geq 2, \quad 2 \leq i \leq r$.
$|F_1 \cap F_0| \geq 1$.

Then G is k-edge colourable.



To apply the lemma we let $r = d_G(v)$.
$e_1, e_2, \ldots, e_r$ are all the edges incident with v.
$F_0 = \{1, 2, \ldots, \Delta + 1\}$.
$|F_i| \geq 2$ for $1 \leq i \leq r$ since if w_i is a neighbour of v in G then $d_{G'}(w_i) \leq \Delta - 1$.

So we can apply Lemma 4 to conclude that G is $\Delta + 1$ colourable. $\square$

Proof of Lemma 4 This is by induction on r.

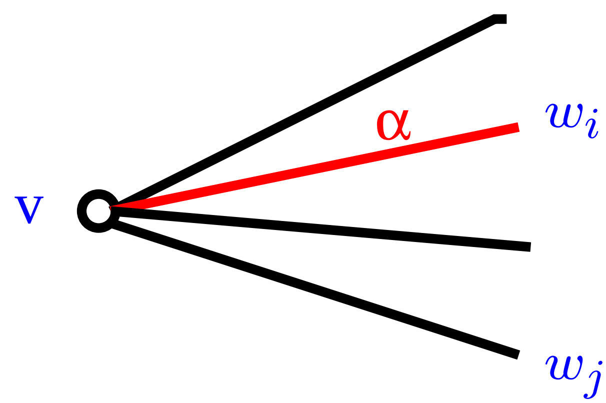
Case r**=1**: we extend the colouring of G^* to G by giving e_1 a colour from $F_0 \cap F_1$.

Inductive Step

Choose $C_1 \subseteq F_0 \cap F_1$ and $C_i \subseteq F_0 \cap F_i$ where

$$|C_1| = 1 \text{ and } |C_i| = 2 \text{ for } 2 \leq i \leq r.$$

SubCase 1: There is a colour α such that α is in exactly **one** of $C_1, C_2, \ldots, C_r$. Suppose $\alpha \in C_i$. Colour e_i with α.

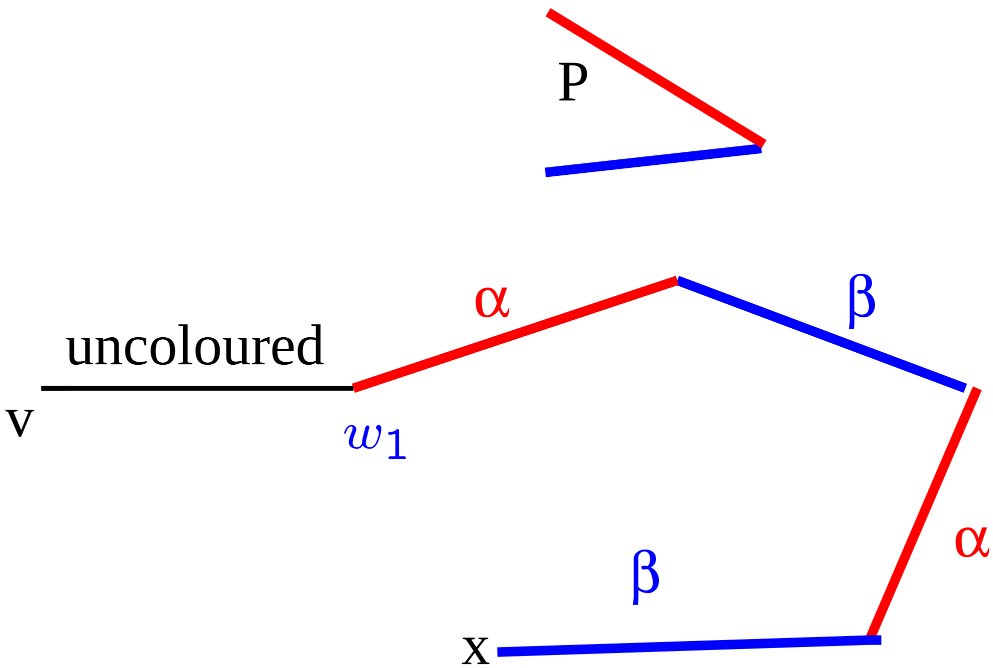


$\alpha \notin C_j$ for $j \neq i$ and so the colours C_j are still missing from v and w_j for $j \neq i$.
We can apply induction for the case $r - 1$ to finish the colouring.

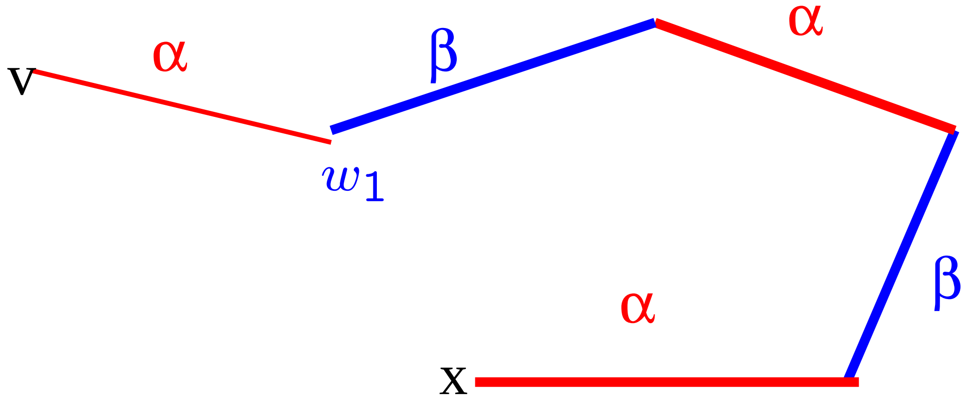
SubCase 2: No colour occurs in exactly one C_i.

There exists a colour $\alpha \in F_0 \setminus \bigcup_{i=1}^{r} C_i$.
($|F_0| \geq k - (\Delta - r) > r$ and $|\bigcup_{i=1}^{r} C_i| < r$.)

Let $C_1 = \{\beta\}$ and let P be the path containing w_1 in the subgraph of G' induced by edges of colour α or β.



Recolour P



Note that $x \neq v$ or w_1 since α, β are both missing at v and β is missing at w_1.

The vertices in the interior of P have the same set of missing colours after the exchange of colours.

Thus at most one $C_i, i \geq 2$ changes (if $x = w_i$) and then by one. We have coloured one more edge, e_1, and so we can again apply induction for the case $r-1$ to finish the colouring.