Network Flows

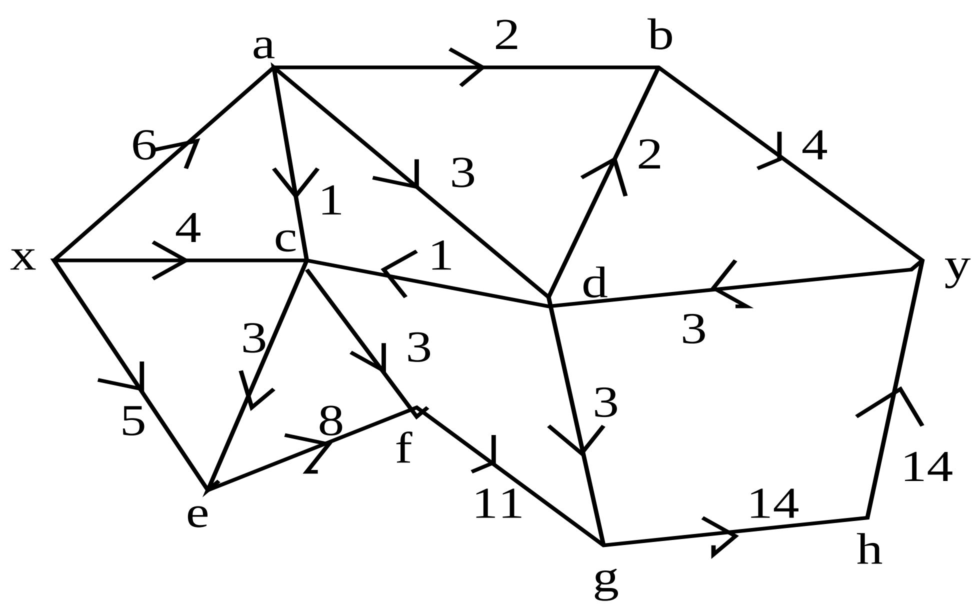
A *Network* is a digraph $D = (V, A)$ plus 2 distinguished vertices, a *source* x and a *sink* y.
Notation: if $f : A \to \boldsymbol{R}$ then for $S, T \subseteq V$,

$$f(S,T) = \sum_{(u,v) \in A \cap (S \times T)} f(u,v)$$

f is a flow from x to y if

$$f(v,V) - f(V,v) = 0$$

for all $v \in V, v \neq x, y$ – *conservation of flow*.



Arc a has *capacity* $c(a) \geq 0$.

A flow is *feasible* if

$$0 \leq f(a) \leq c(a) \qquad\qquad a \in A.$$

Lemma 1 *If f is a flow from x to y then*

$$f(x, V) - f(V, x) = f(V, y) - f(y, V).$$

Proof

$$\begin{aligned} 0 &= f(V, V) - f(V, V) \\ &= [f(x, V) + f(y, V)] - [f(V, x) + f(V, y)] + \\ &\quad + \sum_{v \neq x, y} (f(v, V) - f(V, v)) \\ &= [f(x, V) + f(y, V)] - [f(V, x) + f(V, y)]. \end{aligned}$$

$\square$

$f(x, V) - f(V, x)$ is the *net* flow out of x.

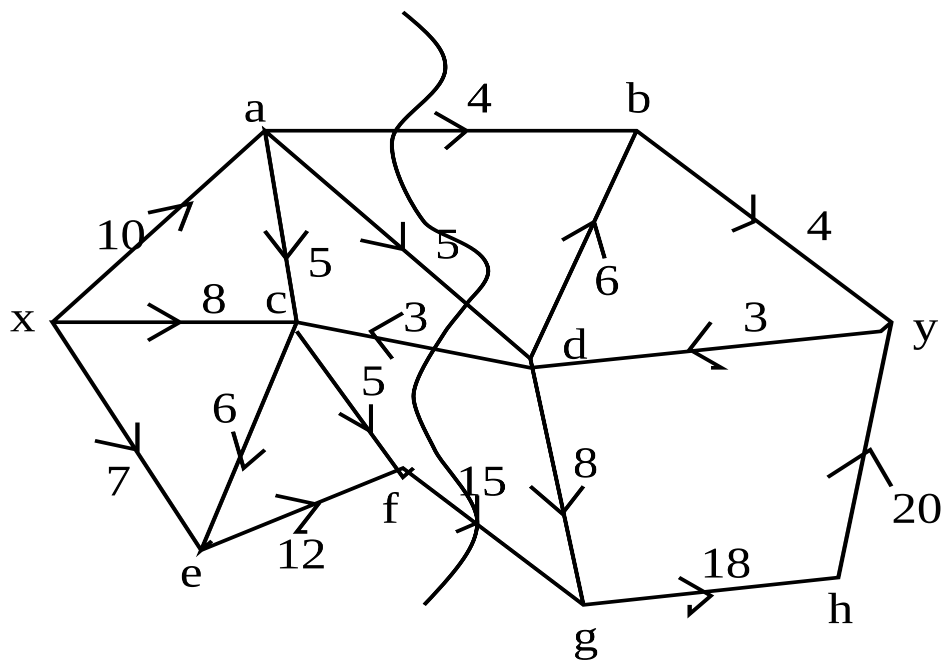
$f(V, y) - f(y, V)$ is the *net* flow into y.

The common value is called the *value* v_f of the flow f.

A feasible flow which maximises v_f is called a *maximum flow*.

Cuts

Let $x \in S \subseteq V$ and $y \in \bar{S} = V \setminus S$. The set of arcs $S : \bar{S} = A \cap (S \times \bar{S})$ is called an x, y *cut*.



S={x,a,c,e,f}: capacity of S:$\bar{S}$ is 4+5+15=24.

$S : \bar{S}$ has *capacity* $c(S, \bar{S})$.

Lemma 2 *If f is a feasible flow and $S : \bar{S}$ is an x, y cut then*

$$v_f \leq c(S : \bar{S}).$$

Proof

$$\begin{aligned} v_f &= f(x, V) - f(V, x) \\ &= \sum_{v \in S} f(v, V) - \sum_{v \in S} f(V, v) \\ &= f(S, S) + f(S, \bar{S}) - f(S, S) - f(\bar{S}, S) \\ &= f(S, \bar{S}) - f(\bar{S}, S) \qquad (1) \\ &\leq c(S : \bar{S}). \end{aligned}$$

□

Flow f *saturates* arc a if $f(a) = c(a)$.

Lemma 3 *If flow f^* and x, y cut $S^* : \bar{S}^*$ are such that*

(i) *f^* saturates every arc of $S^* : \bar{S}^*$.*

(ii) *$f^*(a) = 0$ for every $a \in \bar{S}^* : S^*$.*

then

(a) *$v_{f^*} = c(S^* : \bar{S}^*)$.*

(b) *f^* is a maximum flow.*

(c) *$S^* : \bar{S}^*$ is a minumum capacity cut.*

Proof (a) follows from (i), (ii) and (1). Now let f be any feasible flow and let $S : \bar{S}$ be any x, y cut. Then

$$v_f \leq c(S^* : \bar{S}^*) = v_{f^*} \leq c(S : \bar{S}).$$

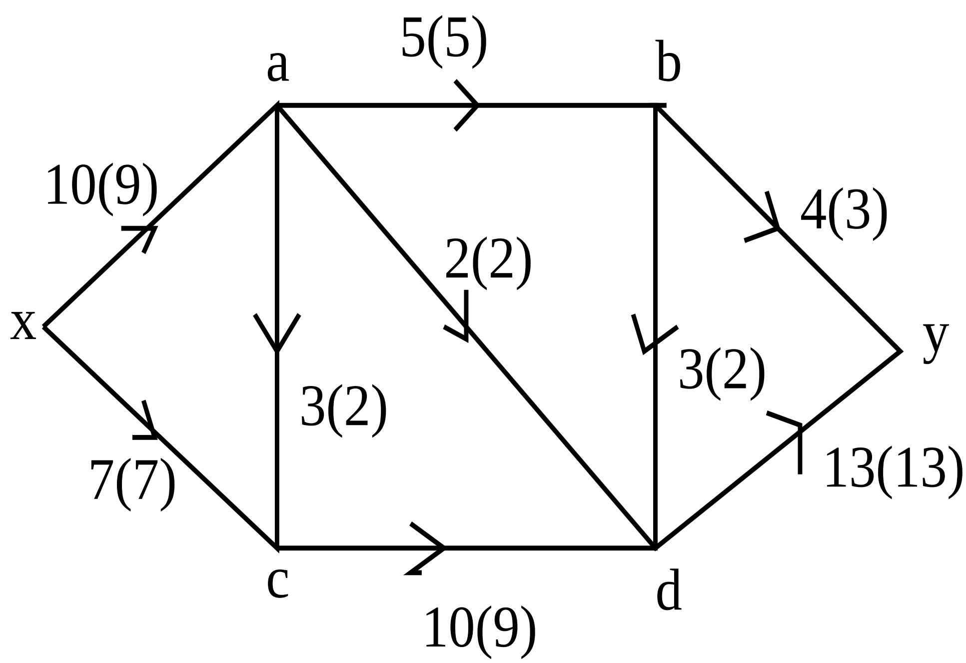
□

f-augmenting paths

Let f be a feasible flow. A path $P = (x_0 = x, x_1, \dots,$ $x_k = y)$ from x to y in the *underlying graph* $G(D)$ is f-augmenting if

$$x_i x_{i+1} \in A \text{ implies that } f(x_i x_{i+1}) < c(x_i x_{i+1}). \quad (2)$$

$$x_{i+1} x_i \in A \text{ implies that } f(x_{i+1} x_i) > 0. \quad (3)$$



x,a,c,d,b,y is f-augmenting

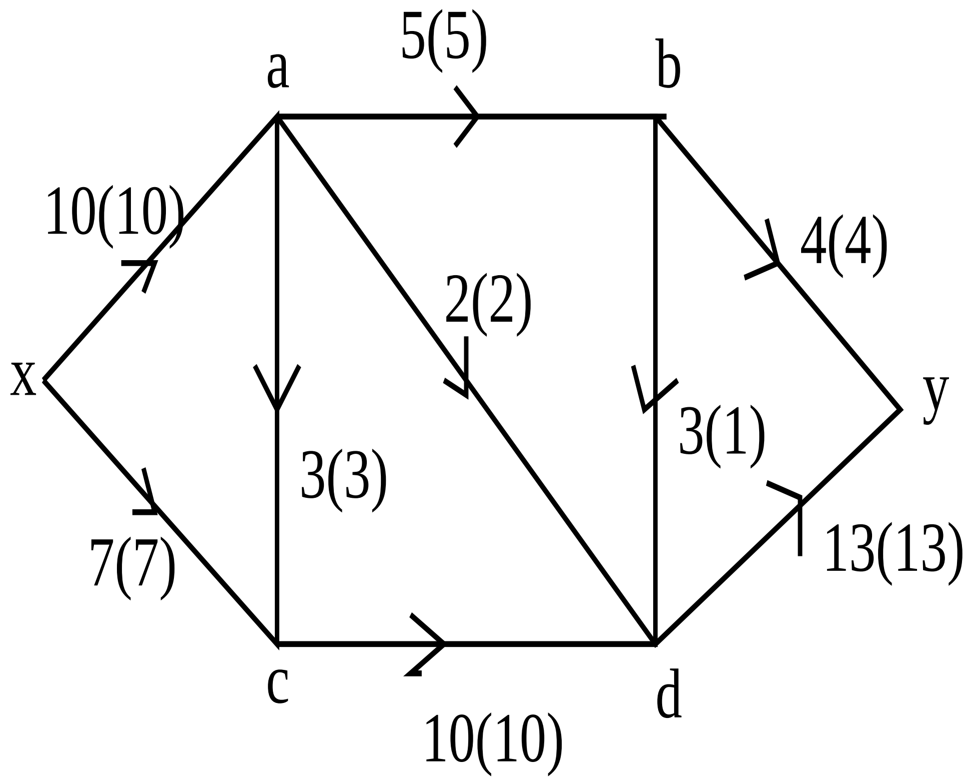
Theorem 1 *f is a maximum flow iff if there are no f-augmenting paths.*

Proof **If:** Suppose $P = (x_0 = x, x_1, \ldots, x_k = y)$ is an f-augmenting path. let

$$\theta = \min \begin{cases} c(x_i x_{i+1}) - f(x_i x_{i+1}) & x_i x_{i+1} \in A \\ f(x_{i+1} x_i) & x_{i+1} x_i \in A \end{cases} \tag{4}$$

Then $\theta > 0$.
Define f' by

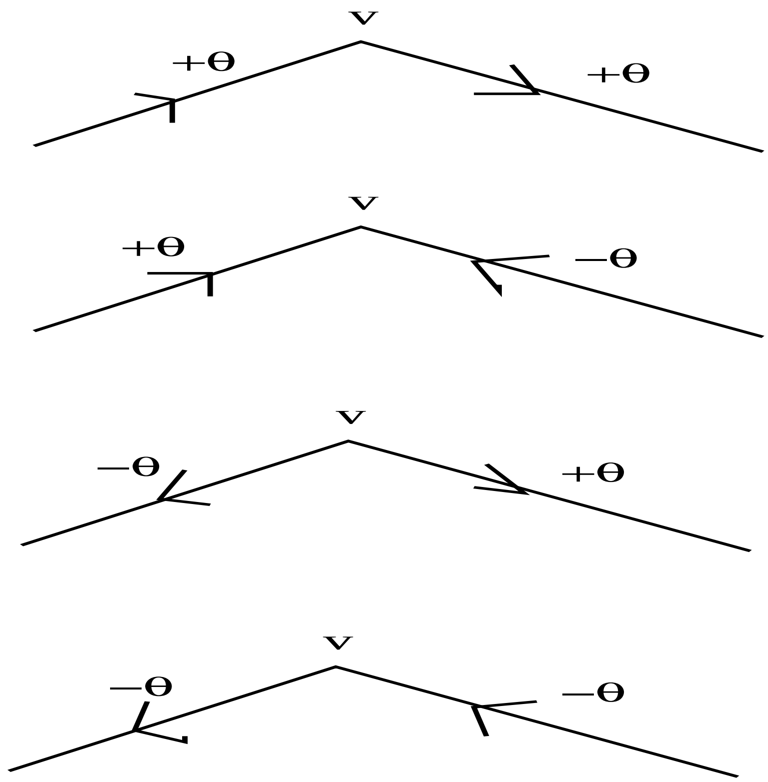
$$f'(a) = \begin{cases} f(x_i x_{i+1}) + \theta & a = x_i x_{i+1} \in A \\ f(x_{i+1} x_i) - \theta & a = x_{i+1} x_i \in A \\ f(a) & \text{otherwise} \end{cases}$$



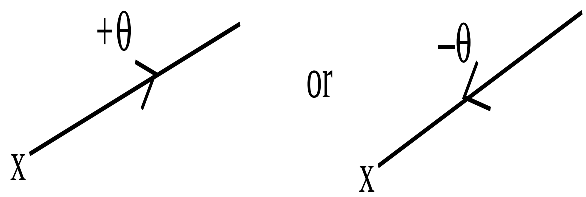
(i) f' is a flow.

$v \notin P \Rightarrow f'(v,V) = f(v,V)$ and $f'(V,v) = f(V,v)$

$v \in P$

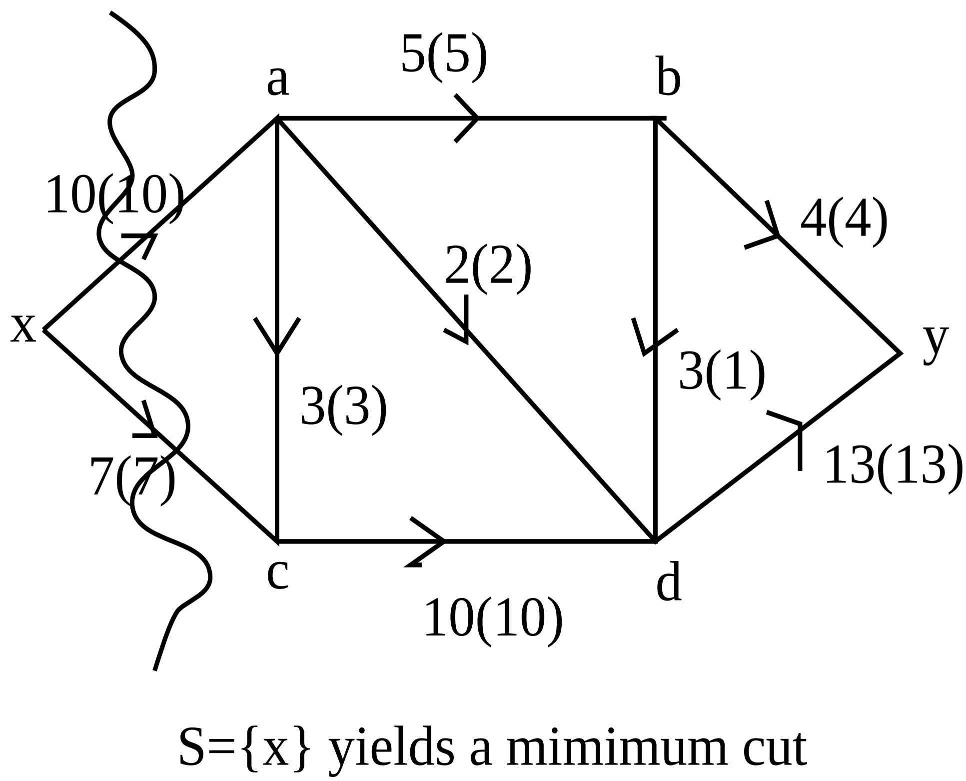


(iii) $v_{f'} = v_f + \theta > v_f$.



Only if: Suppose there are no f-augmenting paths. let

$$S = \{u \in V : \exists \text{ a path } P_u = (x_0 = x, x_1, \ldots, x_k = u) \text{in } G \tag{5}$$



S={x} yields a mimimum cut

Then

(i) $x \in S$ and $y \notin S$

(ii) $a = uv \in S : \bar{S}$ implies $f(a) = c(a)$. If $f(a) < c(a)$ then (P_u, v) satisfies (2),(3) and so $v \in S$ – contradiction.

(iii) $a = vu \in \bar{S} : S$ implies $f(a) = 0$. If $f(a) > 0$ then (P_u, v) satisfies (2),(3) and so $v \in S$ – contradiction.

It follows from Lemma 3 that f is a maximum flow (and $S : \bar{S}$ is a minimum cut).

Max-Flow Min-Cut Theorem

Theorem 2

$$\max_f v_f = \min_S c(S : \bar{S}). \tag{6}$$

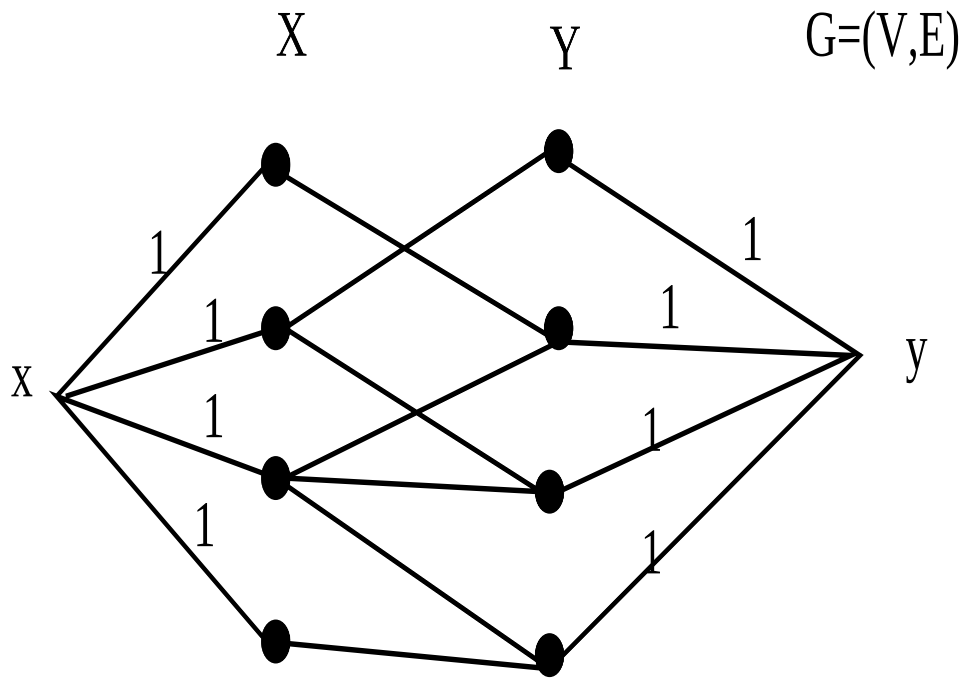
Proof Lemma 2 shows that the LHS of (6) is at most the RHS.

Suppose f is a maximum flow. Let S be as defined in (5). f has no f-augmenting paths and so
$v_f = c(S : \bar{S})$. □

Lemma 4 *If $c(a)$ is an integer for all $a \in A$ then there is a maximum flow with $f(a)$ integer for all $a \in A$.*

Proof Start with the feasible flow $f = 0$. Repeatedly find flow augmenting paths until a maximum flow is reached. We can argue inductively that f stays integer throughout. This is because θ of (4) will be integer if f and c are. $\square$

Alternate proof of Hall's Theorem



$m = |X| \leq |Y|$.
Let

$$c(a) = \begin{cases} 1 & a = xu, u \in X \\ 1 & a = vy, v \in Y \\ \infty & a \in E \end{cases}$$

An integral flow f from x to y defines a matching

$$M = \{uv \in E : \ f(uv) = 1\},$$

and conversely.

Let $S : \bar{S}$ be an x, y cut and let

$$S_1 = S \cap X,\ S_2 = S \cap Y.$$

If $\exists u \in S_1$ and $v \in Y \setminus X_2$ such that $uv \in E$ then

$$c(S : \bar{S}) \geq c(uv) = \infty.$$

So

$$c(S : \bar{S}) < \infty \text{ iff } N(S_1) \subseteq S_2.$$

In which case

$$c(S : \bar{S}) = (|X| - |S_1|) + |S_2|.$$

By the Max-Flow Min-Cut Theorem

$$\begin{aligned} \max\{|M|\} &= \min_{\substack{S_1 \subseteq X \\ N(S_1) \subseteq S_2 \subseteq Y}} (|X| - |S_1|) + |S_2| \\ &= \min_{S_1 \subseteq X} (|X| - |S_1|) + |N(S_1)| \end{aligned}$$

Thus there exists a matching of size $|X|$ iff

$$|X| - |S_1| + |N(S_1)| \geq |X|$$

for all $S_1 \subseteq X$, which is Hall's theorem.

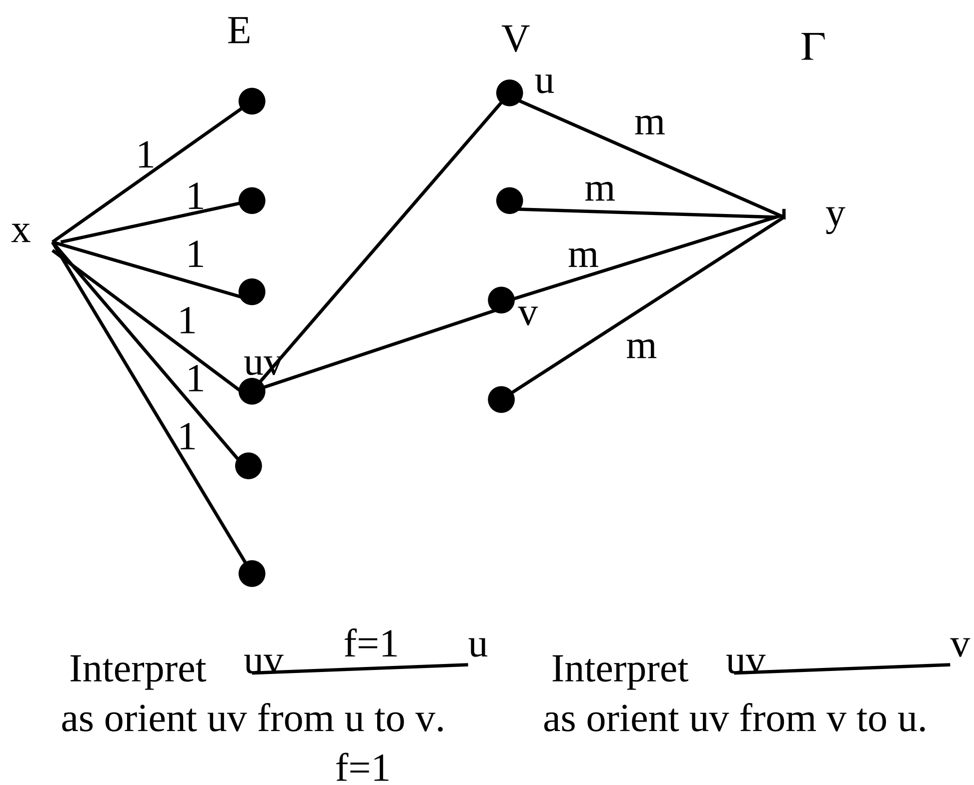
A graph G is m-orientable if there is an orientation D of G with $\delta^+(D) \geq m$. ($\delta^+(D) = \min\{d^+(v) : v \in V\}$).

For $S \subseteq V$ let $\imath(S)$ denote the number of edges of G with at least one end in S.

Theorem 3 *G is m-orientable iff $\imath(S) \geq m|S|$ for all $S \subseteq V$.*

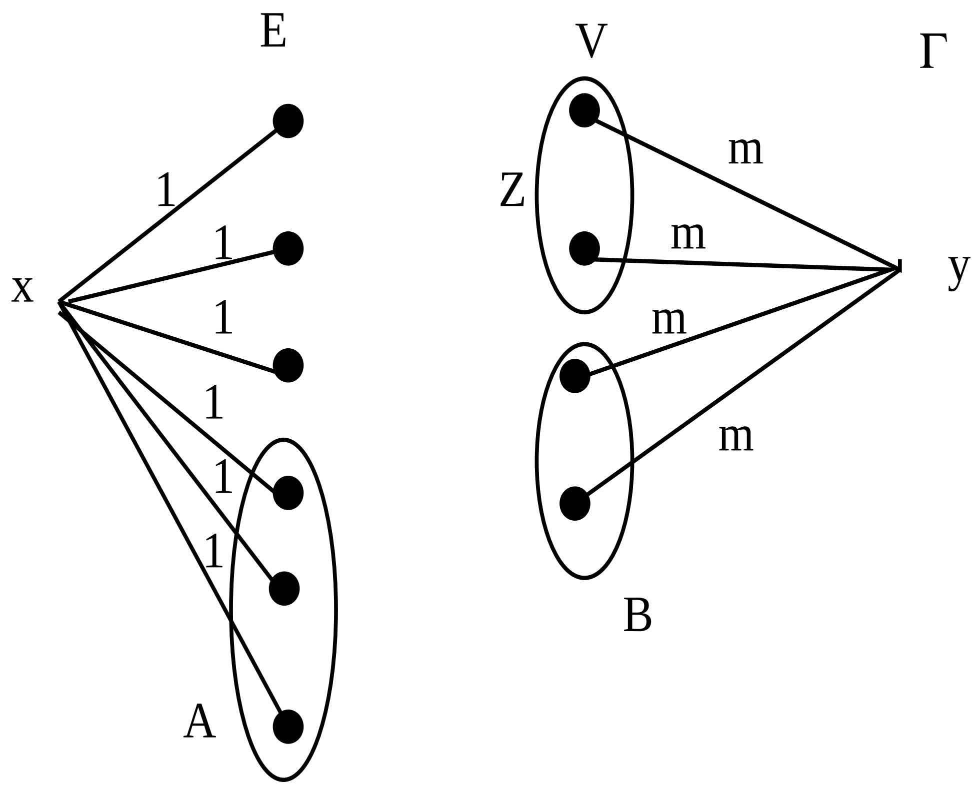
Proof **Only if:** Suppose that D is an orientation of G with $\delta^+ \geq m$. Then

$$\imath(S) \geq \sum_{v \in S} d^+(S) \geq m|S|.$$



G is m-orientable iff there exists a flow of value $m|V|$.

Suppose the maximum flow value is $< m|V|$. Let $S : \bar{S}$ be a minimum cut in Γ. Let $A = S \cap E$ and $B = S \cap V$.



There are no edges from A to Z in Γ else $c(S : \bar{S}) = \infty$. So

$$\begin{aligned} \imath(Z) &\leq |E| - |A| \\ |E| - |A| + m|B| &< m|V| \end{aligned}$$

and $\imath(Z) < m|Z|$. □

Menger's Theorems

In the following $x, y \in V$.

Theorem 4 *The maximum number of arc disjoint directed paths joining x and y in a digraph D equals the minimum number of arcs whose deletion destroys all directed x, y paths.*

Theorem 5 *The maximum number of internally vertex disjoint directed paths joining x and y in a digraph D equals the minimum number of vertices ($\neq x, y$) whose deletion destroys all directed x, y paths.*

Theorem 6 *The maximum number of edge disjoint paths joining x and y in a graph G equals the minimum number of edges whose deletion destroys all x, y paths.*

Theorem 7 *The maximum number of internally vertex disjoint paths joining x and y in a graph D equals the minimum number of vertices ($\neq x, y$) whose deletion destroys all x, y paths.*

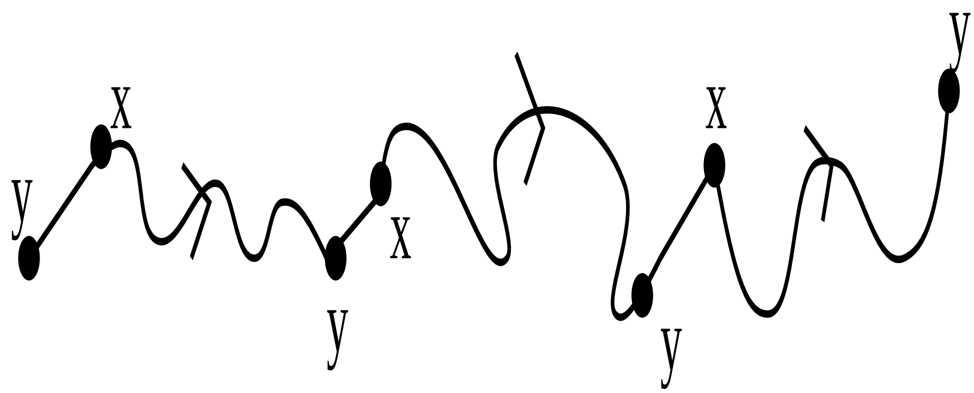
Lemma 5 *Let N be a network in which each arc has capacity 1. Let f^* be a maximum flow and $S^* : \bar{S}^*$ a minimum cut.*

(a) v_{f^*} is the maximum number m_1^*, of arc disjoint directed x, y paths.

(b) $c(S^* : \bar{S}^*)$ is the minimum number m_2^* of arcs whose deletion destroys all directed x, y paths.

(a) If $P_1, P_2, \ldots, P_{m_1^*}$ is a set of arc disjoint directed x, y paths then we can send one unit of flow along each path. Thus $v_{f^*} \geq m_1^*$.

To prove $v_{f^*} \leq m_1^*$ delete all arcs with $f^*(a) = 0$ to obtain arc set A^*. Note that $f^*(a) = 1$ for $A \in A^*$. Add v_{f^*} yx arcs. The digraph $D^* = (V, A^*)$ has an Euler tour. Deleting the yx edges from the tour yields v_{f^*} arc disjoint directed x, y paths.



(b) Let $S : \bar{S}$ be an x, y cut in N. $S : \bar{S}$ meets every x, y path and so deleting $S : \bar{S}$ destroys all x, y paths and $c(S : \bar{S}) = |S : \bar{S}| \geq m_2^*$.

On the other hand, if X is any set of arcs which meet every x, y path, let $S = \{v : \ v$ is reachable from x by a directed path in $D - X\}$. Then $y \in \bar{S}$ and $X \supseteq S : \bar{S}$. (If there is an arc $uv \notin X$, $u \in S, v \in \bar{S}$ then v is reachable from x in $D - X$, contradiction.) Thus $|X| \geq c(S : \bar{S})$ which implies m_2^* is at least the minimum capacity of a cut. □

Theorem 4 follows from the above lemma and the Max-Flow Min-Cut theorem.

Lemma 6 *Let*

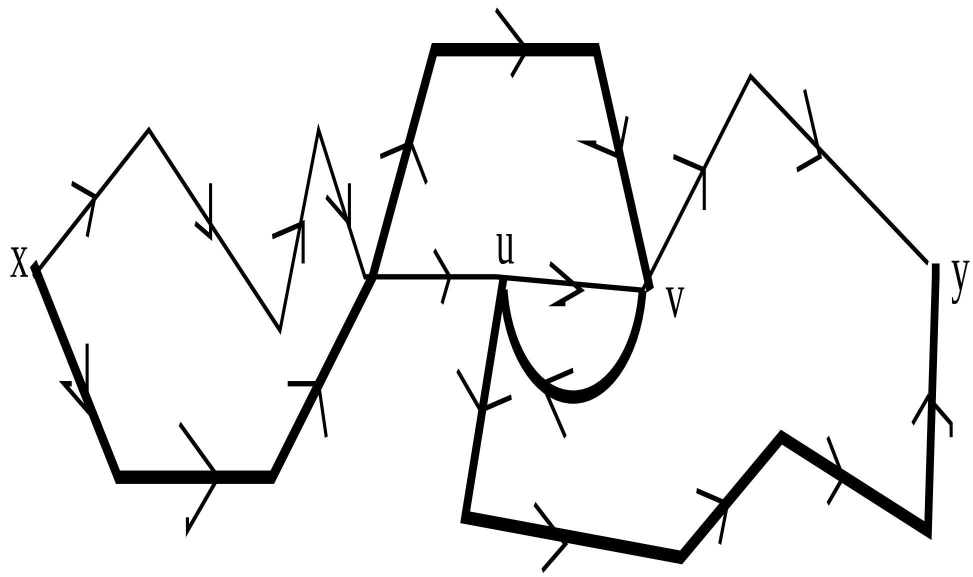
m_1 *be the maximum number of arc disjoint* x, y *directed paths in* $D(G)$*.*

m_2 *be the maximum number of arc disjoint* x, y *directed paths in* $D(G)$ *such that*

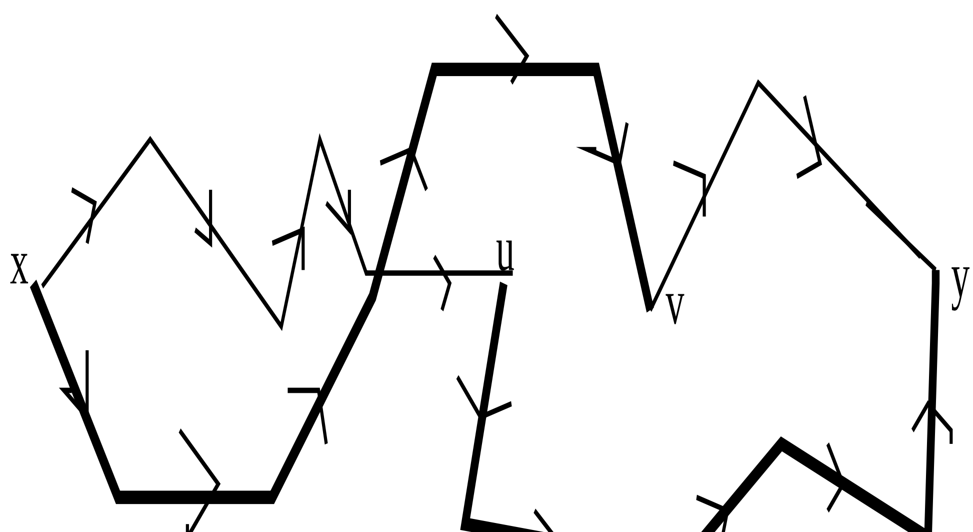
$$\textit{at most one of } uv, vu \textit{ can be used as an edge in the set of paths.} \quad (7)$$

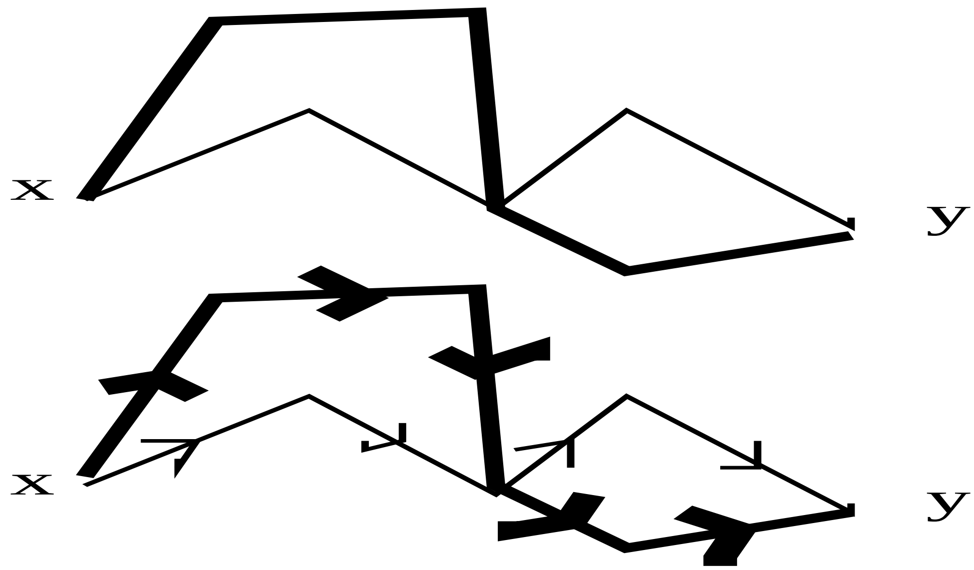
Then $m_1 = m_2$*.*

Proof Clearly $m_1 \geq m_2$. For the converse, let $P_1, P_2, \ldots, P_{m_1}$ be a collection of arc disjoint x, y directed paths and assume that $\sum |P_i|$ is as small as possible. We claim that (7) holds.



We can reduce $\sum |P_i|$ by removing the uv and vu.





Proof of Theorem 6.

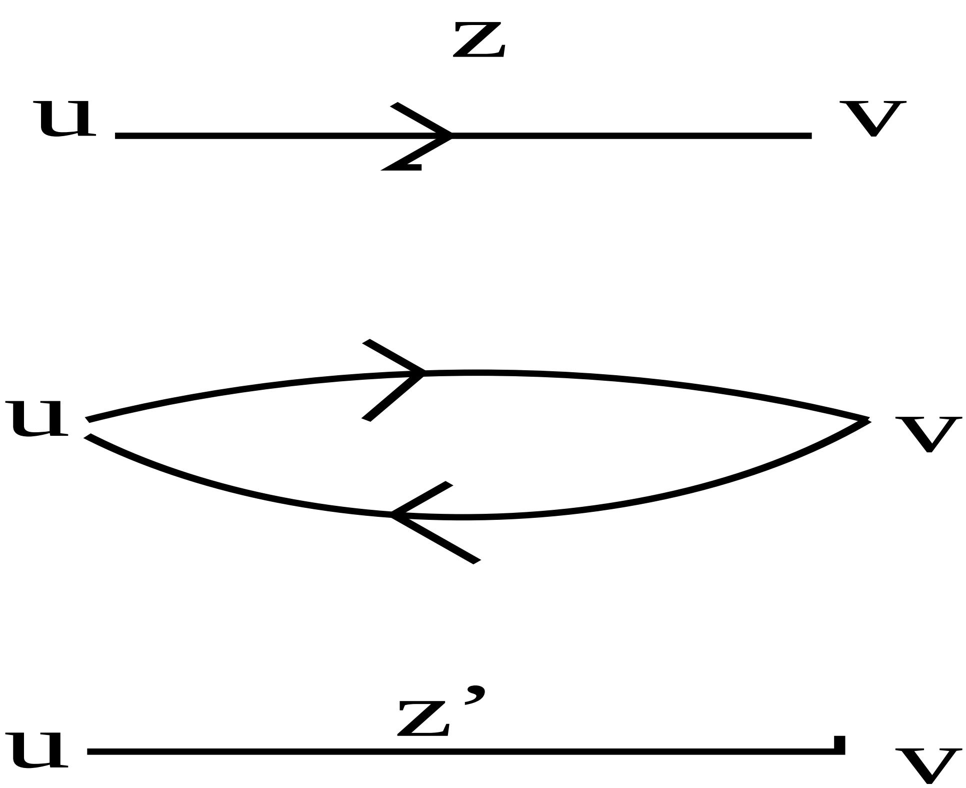
$m =$ max. number of edge disjoint x, y paths in G

$= m_2$ of Lemma 6

$= m_1$ of Lemma 6

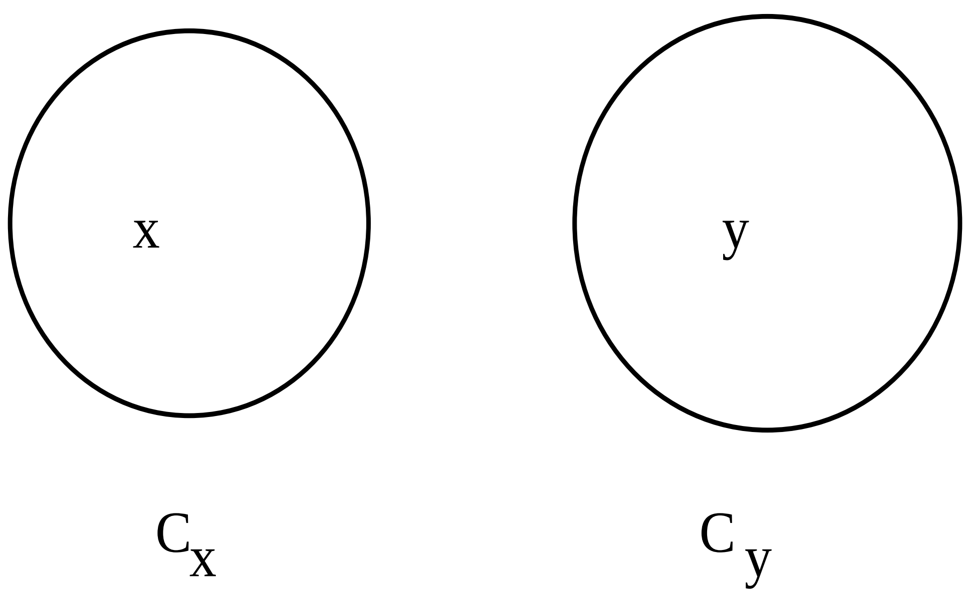
$= \widehat{m}_1$(the minimum number of arcs whose deletion destroys all directed x, y paths in $G(D)$ by Theorem 4)

$\geq m' =$ minimum number of edges whose deletion destroys all x, y paths in G.



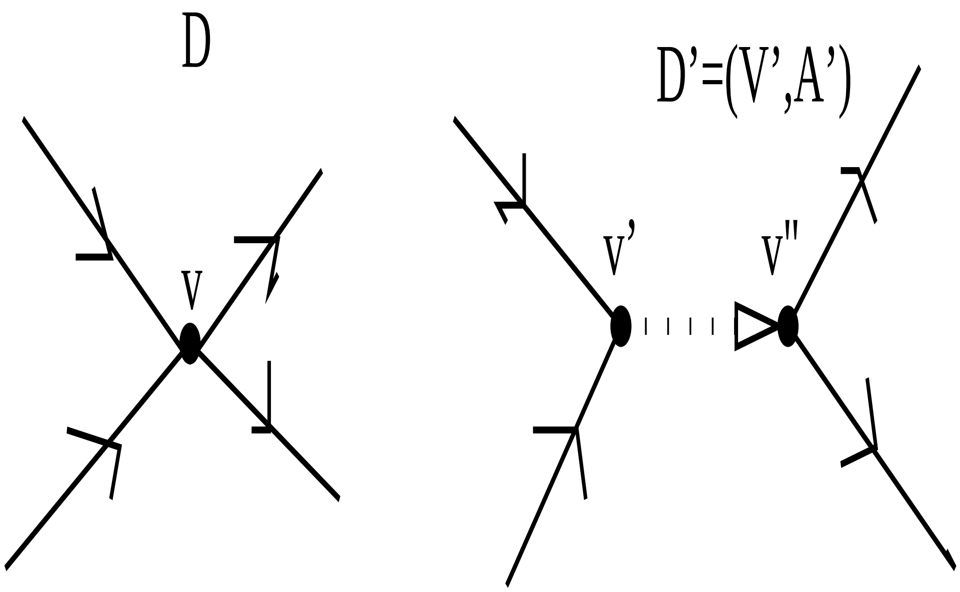
If Z covers all x, y paths in $D(G)$ then Z' covers all x, y paths in G.

We finish by showing that $m' \geq \hat{m}_1$. Suppose that the deletion of $X, |X| = m'$ destroys all x, y paths in G. X is minimal with this property. So $G - X$ has two components.



Let $Y = \{uv : \; uv \in X, u \in C_x, v \in C_y\}$. Then $|X| = |Y|$ and there are no directed x, y paths in $D(G) - Y$. Thus $m' \geq \hat{m}_1$. $\square$

Proof of Theorem 5



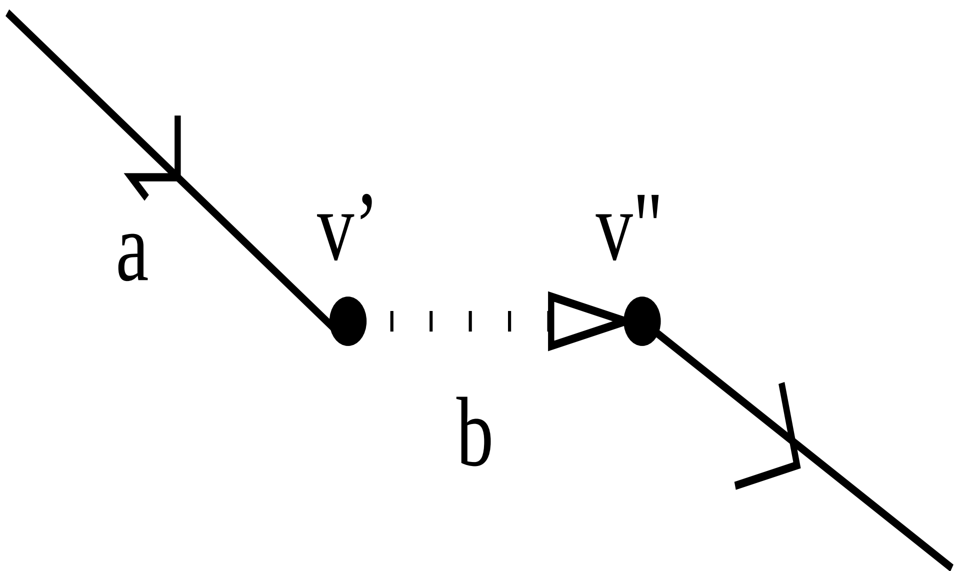
Each vertex v of D becomes an arc a_v of D'. For $S \subseteq V$ let $A_S = \{a_v : \ v \in S\}$.

(a) In the transformation $D \to D'$ node disjoint paths correspond to arc disjoint paths.

(b)

(i) Z covers all directed x, y paths in D implies A_Z covers all directed x, y paths in D'.

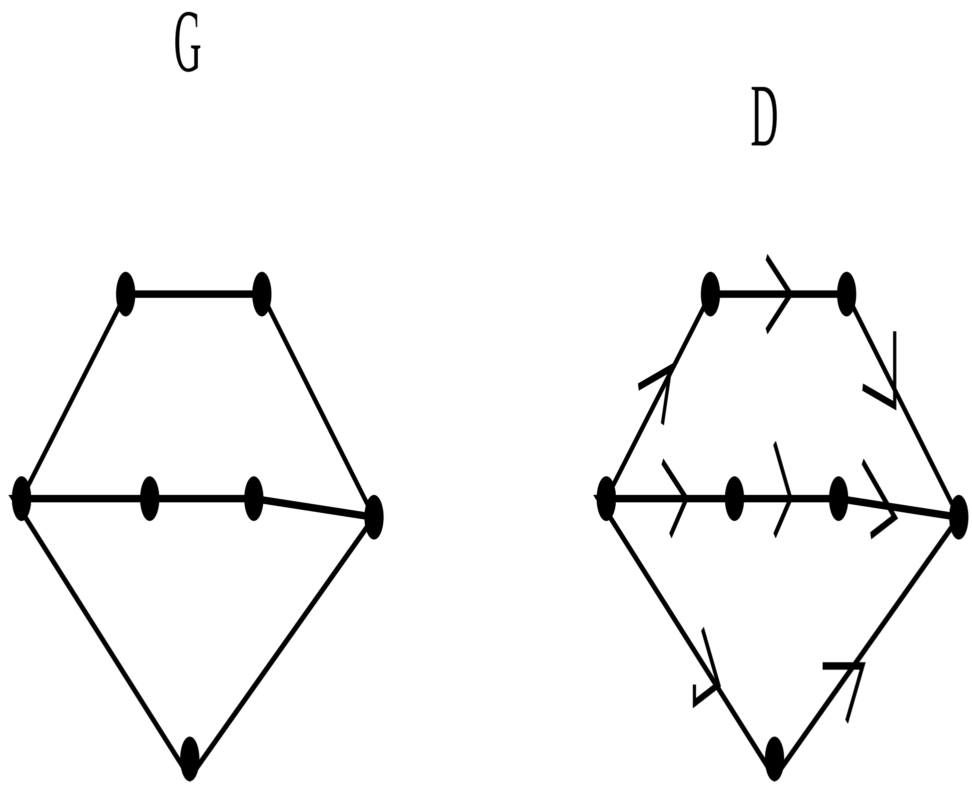
(ii) Y covers all directed x, y paths in D', Y has as few arcs as possible, then we can assume $Y \subseteq A_Z$.



(Can always replace a by b.)

Proof of Theorem 7

Node disjoint paths in G map to node disjoint paths in $G(D)$.



$X \subseteq V$ covers all x, y paths in G iff X covers all directed x, y paths in D. $\square$