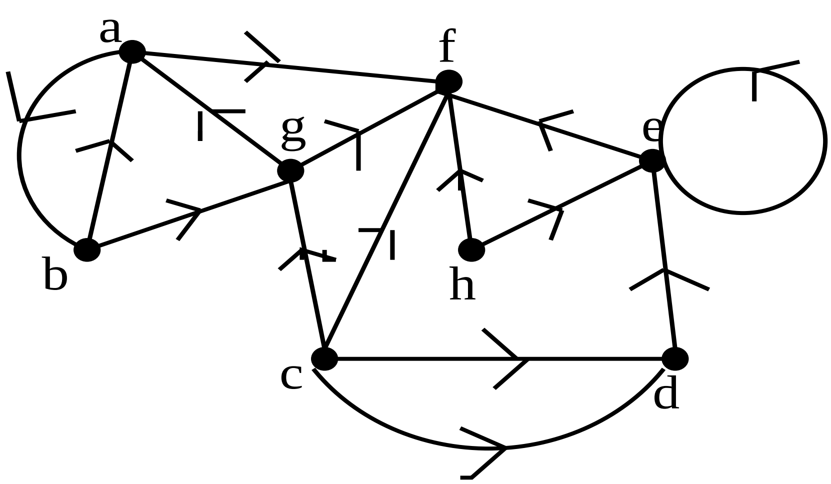
Directed graphs

Digraph $D = (V, A)$.
V={vertices}, A={arcs}

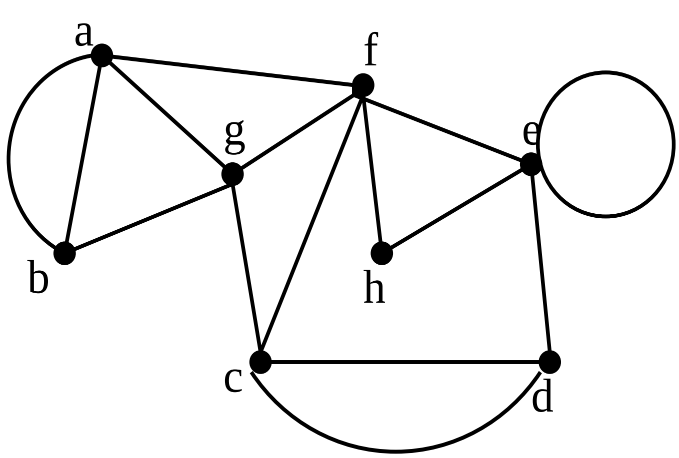


V={a,b,...,h}, A={(a,b),(b,a),...}

(2 arcs with endpoints (c,d))

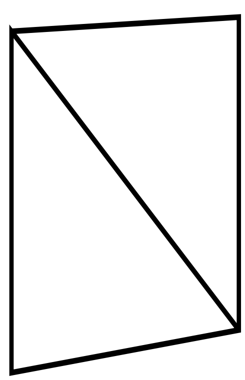
Thus a digraph is a graph with oriented edges.
D is *strict* if there are no loops or repeated edges.

Digraph D: $G(D)$ is the *underlying* graph obtained by replaced each arc (a, b) by an edge $\{a, b\}$.

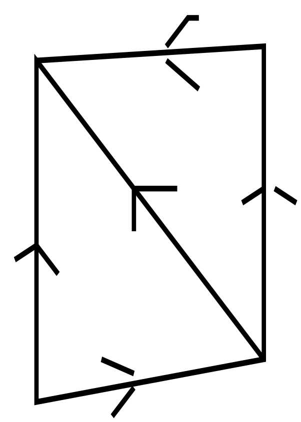


The graph underlying the digraph on previous slide

Graph G: an orientation of G is obtained by replacing each edge $\{a, b\}$ by (a, b) or (b, a).



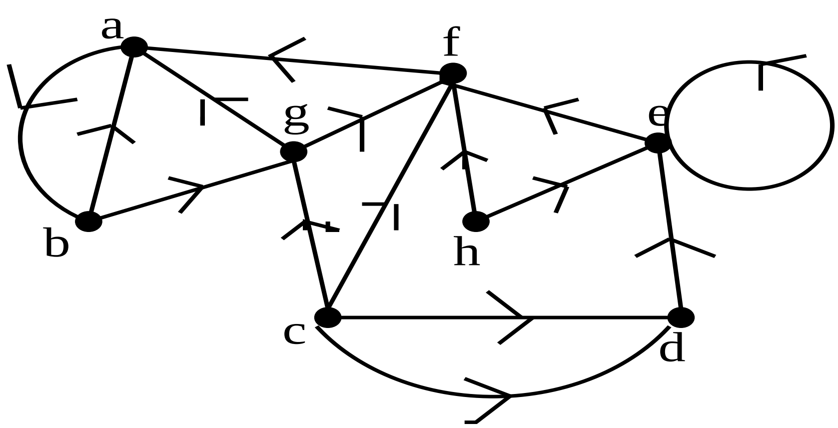
G



Orientation of G

There are $2^{|E|}$ distinct orientations of G.

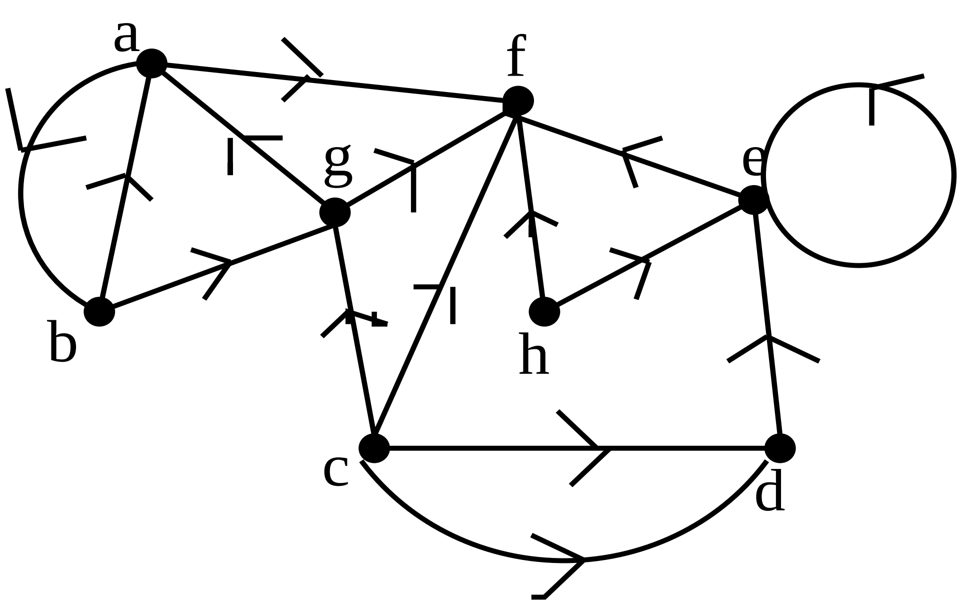
Walks, trails, paths, cycles now have directed counterparts.



Directed Walk: (c,d,e,f,a,b,g,f).
Directed Path: (a,b,g,f).
Directed Cycle: (g,a,b,a)

(e,f,g,a) is not a directed walk -- there is no arc (f,g).

The *indegree* $d_D^-(v)$ of vertex v is the number of arcs $(x, v), x \in V$. The *outdegree* $d_D^+(v)$ of vertex v is the number of arcs $(v, x), x \in V$.



	a	b	c	d	e	f	g	h
d^+	2	2	4	1	2	0	2	2
d^-	2	1	0	2	3	5	2	0

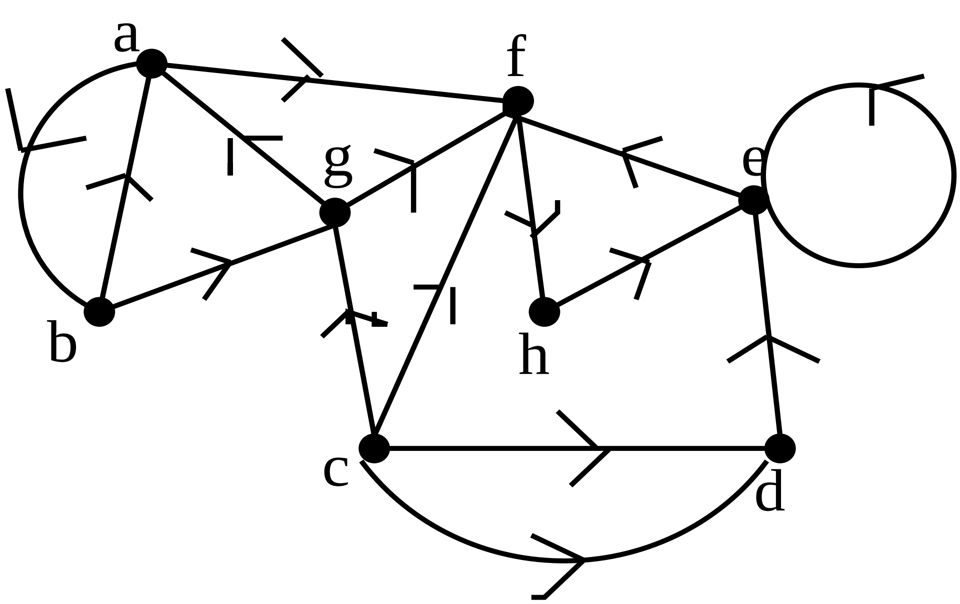
Note that since each arc contributes one to a vertex outdegree and one to a vertex indegree,

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |A|.$$

Strong Connectivity or Diconnectivity

Given digraph D we define the relation $\sim$ on V by $v \sim w$ iff there is a *directed* walk from v to w and a directed walk from w to v.

This is an equivalence relation (proof same as directed case) and the equivalence classes are called *strong components* or *dicomponents*.



Here the strong components are

$$\{a, b, g\}, \{c\}, \{d\}, \{e, f, h\}.$$

A graph is *strongly connected* if it has one strong component i.e. if there is a directed walk between each pair of vertices.

For a set $S \subseteq V$ let

$$N^+(S) = \{w \notin S : \ \exists v \in S \text{ s.t.} (v, w) \in A)\}.$$

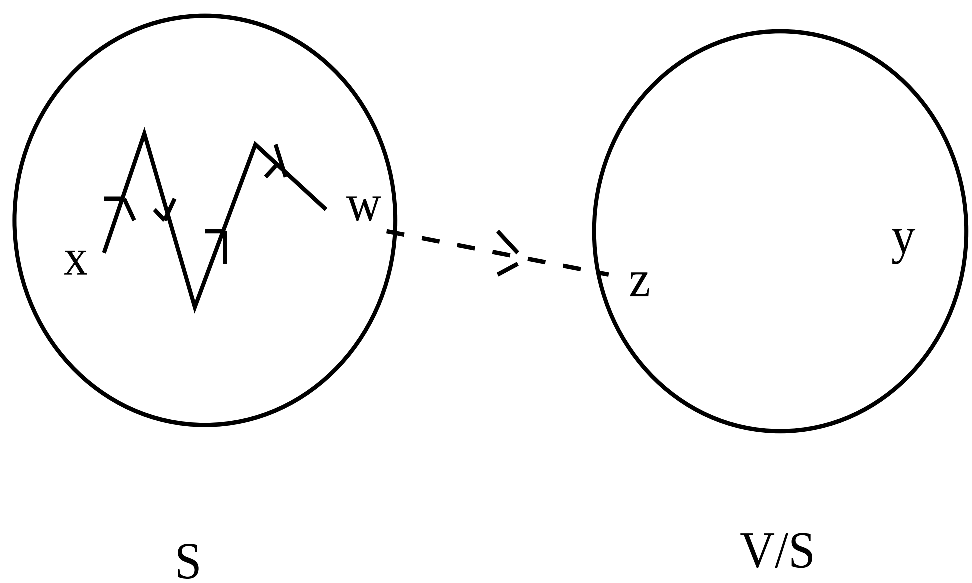
$$N^-(S) = \{w \notin S : \ \exists v \in S \text{ s.t.} (w, v) \in A)\}.$$

Theorem 1 *D is strongly connected iff there does not exist $S \subseteq V$, $S \neq \emptyset, V$ such that $N^+(S) = \emptyset$.*

Proof **Only if**: suppose there is such an S and $x \in S$, $y \in V \setminus S$ and suppose there is a directed walk W from x to y. Let $(v_1 = x, v_2, \ldots, v_k = y)$ be the sequence of vertices traversed by W. Let v_i be the first vertex of this sequence which is not in S. Then $v_i \in N^+(S)$, contradiction, since arc (v_{i-1}, v_i) exists.

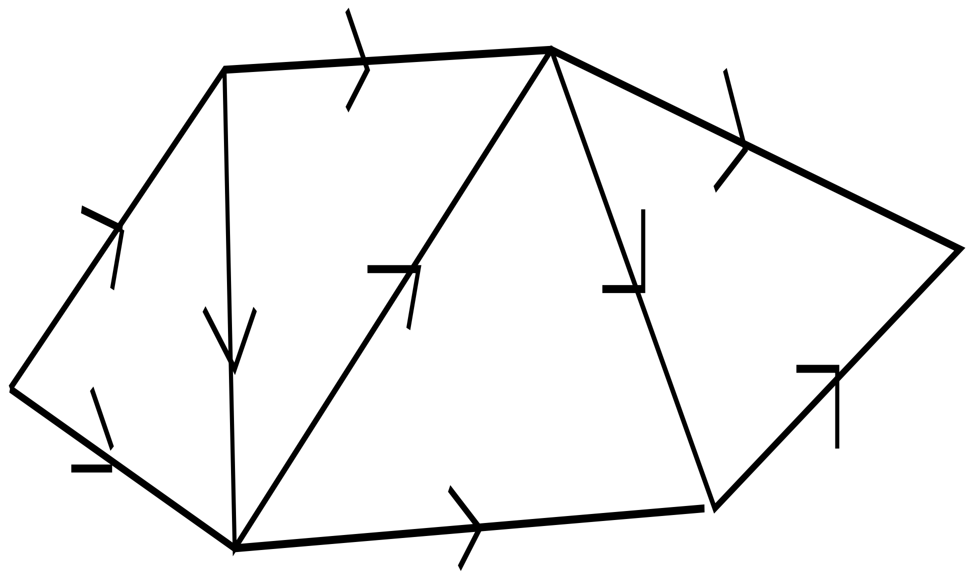
If: suppose that D is not strongly connected and that there is no directed walk from x to y. Let $S = \{v \in V : \exists$ a directed walk from x to $v\}$.

$S \neq \emptyset$ as $x \in S$ and $S \neq V$ as $y \notin S$.



Then $N^+(S) = \emptyset$. If $z \in N^+(S)$ then there exists $w \in S$ such that $(w, z) \in A$. But then since $w \in S$ there is a directed walk from x to w which can be extended to z, contradicting the fact that $z \notin S$. $\square$

A *Directed Acyclic Graph* (DAG) is a digraph without any directed cycles.



Lemma 1 *If D is a DAG then D has at least one* source *(vertex of indegree 0) and at least one* sink *(vertex of outdegree 0).*

Proof Let $P = (v_1, v_2, \ldots, v_k)$ be a directed path of maximum length in D. Then v_1 is a source and v_k is a sink.

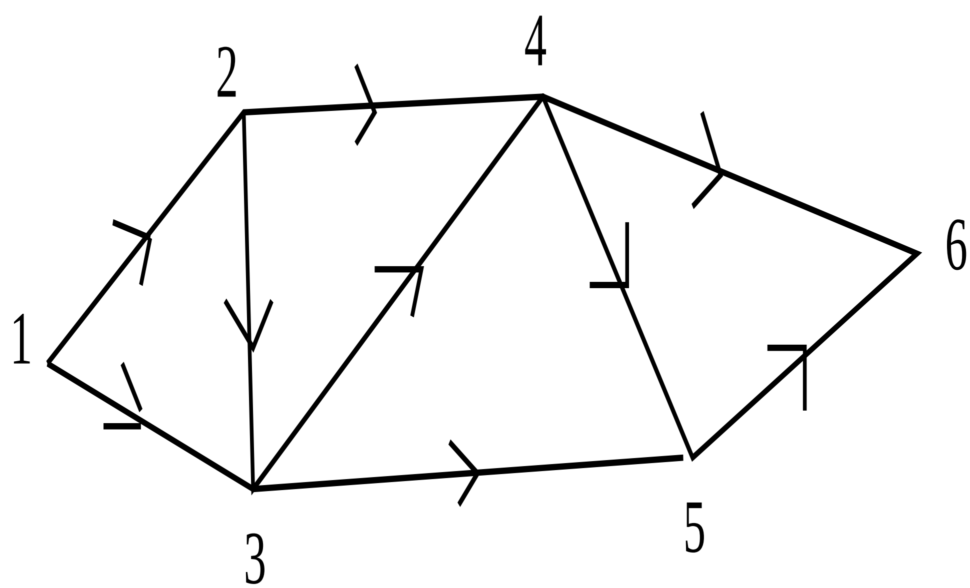
Suppose for example that there is an edge xv_1. Then either

(a) $x \notin \{v_2, v_3, \ldots, v_k\}$. But then (x, P) is a longer directed path than P – contradiction.

(b) $x = v_i$ for some $i \neq 1$ and D contains the cycle $v_1, v_2, \ldots, v_i, v_1$. $\square$

A *topological ordering* $v_1, v_2, \ldots, v_\nu$ of the vertex set of a digraph D is one in which

$$v_i v_j \in A \text{ implies } i < j.$$



Theorem 2 *D has a topological ordering iff D is a DAG.*

Proof **Only if:** Suppose there is a topological ordering and a directed cycle $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. Then

$$i_1 < i_2 < \cdots < i_k < i_1$$

which is absurd.

if: By induction on ν. Suppose that D is a DAG. The result is true for $\nu = 1$ since D has no loops. Suppose that $\nu > 1$, v_ν is any sink of D and let $D' = D - v_\nu$.

D' is a DAG and has a topological ordering $v_1, v_2, \ldots, v_{\nu-1}$, induction. $v_1, v_2, \ldots, v_\nu$ is a topological ordering of D. For if there is an edge $v_i v_j$ with $i > j$ then (i) it cannot be in D' and (ii) $i \neq \nu$ since v_ν is a sink. $\square$

Theorem 3 *Let $G = G(D)$. Then D contains a directed path of length $\chi(G) - 1$.*

Proof Let $D = (V, A)$ and $A' \subseteq A$ be a *minimal* set of edges such that $D' = D - A$ is a DAG.

Let k be the length of the longest directed path in D'.

Define $c(v)$=length of longest path from v in D'. $c(v) \in \{0, 1, 2, \ldots, k\}$. We claim that $c(v)$ is a proper colouring of G, proving the theorem.

Note first that if D' contains a path $P = (x_1, x_2, \ldots, x_k)$ then

$$c(x_1) \geq c(x_k) + k - 1. \qquad (1)$$

(We can add the longest path Q from x_k to P to create a *path* (P, Q). This uses the fact that D' is a DAG.)

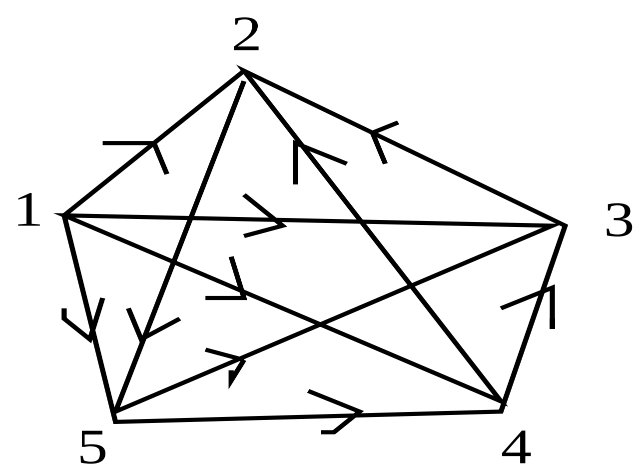
Suppose c is not a proper colouring of G and there exists an edge $vw \in G$ with $c(v) = c(w)$. Suppose $vw \in A$ i.e. it is directed from v to w.

Case 1: $vw \notin A'$. (1) implies $c(v) \geq c(w) + 1$ – contradiction.

Case 2: $vw \in A'$. There is a cycle in $D' + vw$ which contains vw, by the minimality of A'. Suppose that C has $\ell \geq 2$ edges. Then (1) implies that $c(w) \geq c(v) + \ell - 1$. $\square$

Tournaments

A tournament is an orientation of a complete graph K_n.



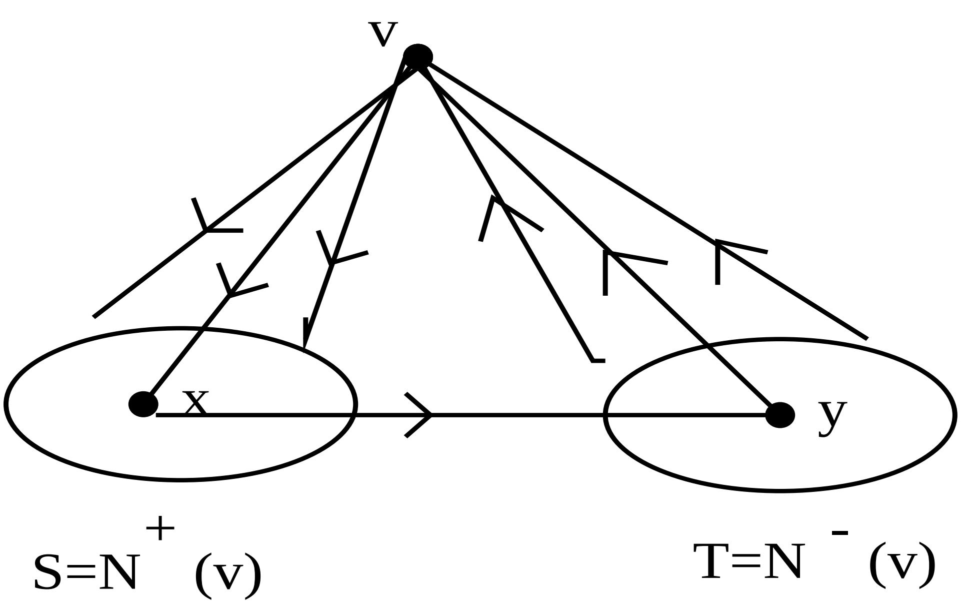
1,2,5,4,3 is a directed Hamilton Path

Corollary 1 *A tournament T contains a directed Hamilton path.*

Proof $\chi(G(T)) = n$. Now apply Theorem 3. □

Theorem 4 *If D is a strongly connected tournament with $\nu \geq 3$ then D contains a directed cycle of size k for all $3 \leq k \leq \nu$.*

Proof By induction on k.
$k = 3$.
Choose $v \in V$ and let $S = N^{+}(V)$, $T = N^{-}(v) = V \setminus (S \cup \{v\})$.

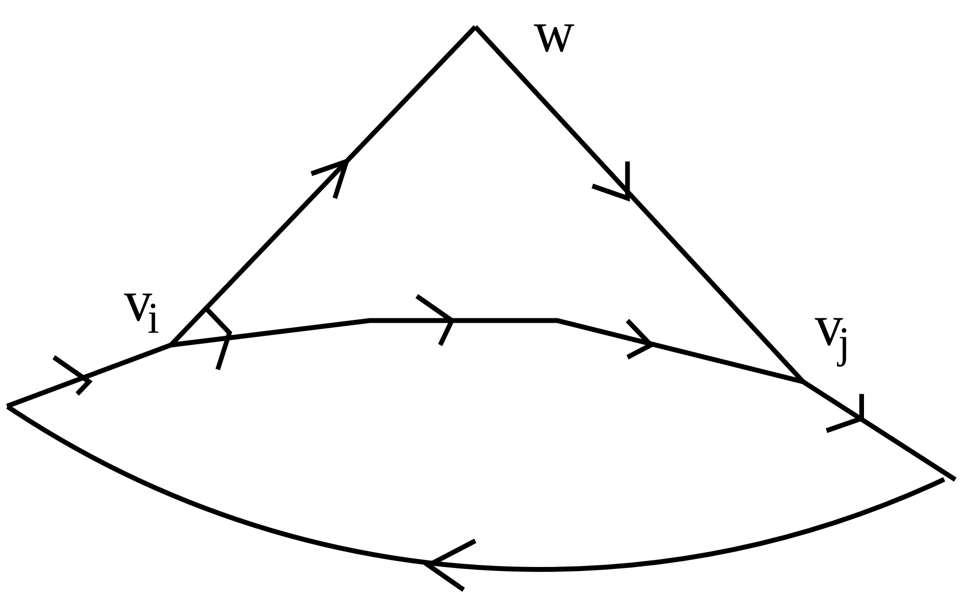


$S \neq \emptyset$ since D is strongly connected. Similarly, $S \neq V \setminus \{v\}$ else $N^{+}(V \setminus \{v\}) = \emptyset$.

Thus $N^{+}(S) \neq \emptyset$. $v \notin N^{+}(S)$ and so $N^{+}(S) = T$. Thus $\exists x \in S, y \in T$ with $xy \in A$.

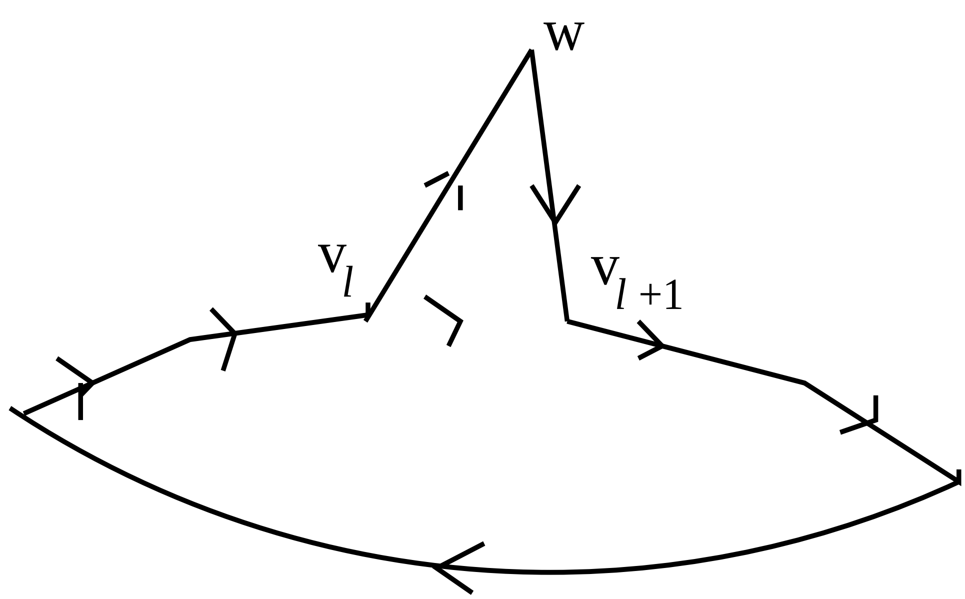
Suppose now that there exists a directed cycle $C = (v_1, v_2, \ldots, v_k, v_1)$.

Case 1: $\exists w \notin C$ and $i \neq j$ such that $v_i w \in A$, $w v_j \in A$.



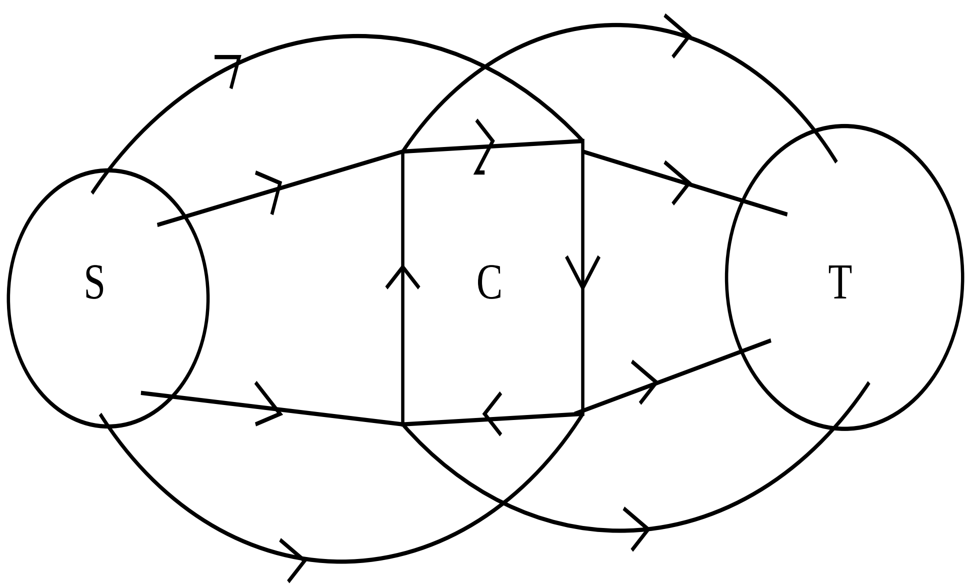
It follows that there exists ℓ with $v_\ell w \in A$, $wv_{\ell+1} \in A$.

$C' = (w, v_{\ell+1}, \ldots, v_\ell, v_1, \ldots, v_\ell, w)$ is a cycle of length $k+1$.



Case 2 $V \setminus C = S \cup T$ where

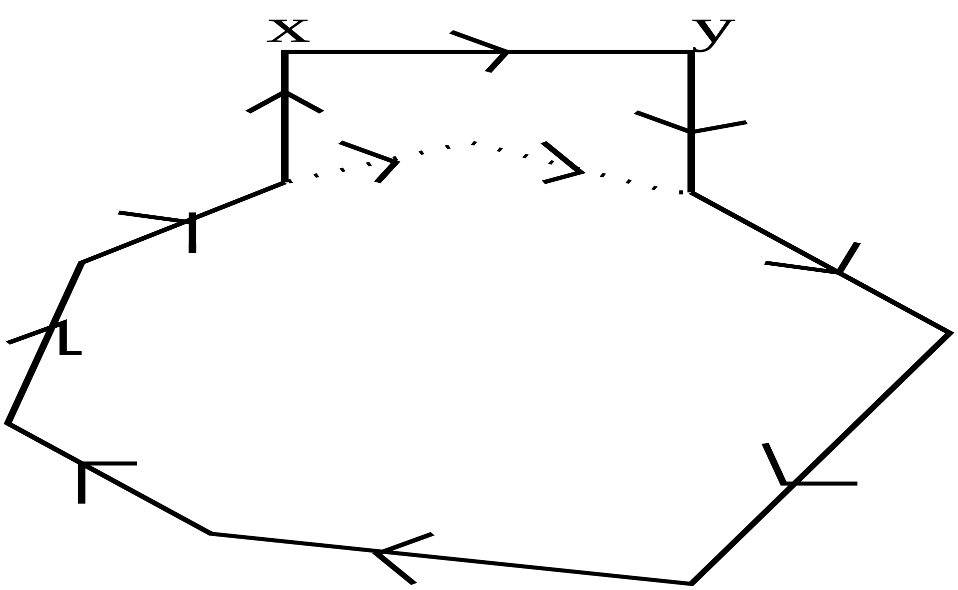
$$w \in S \quad \text{implies} \quad wv_i \in A,\ 1 \leq i \leq k.$$
$$w \in T \quad \text{implies} \quad v_iw \in A,\ 1 \leq i \leq k.$$



$S = \emptyset$ implies $T = \emptyset$ (and C is a Hamilton cycle) or $N^+(T) = \emptyset$.
$T = \emptyset$ implies $N^+(C) = \emptyset$.

Thus we can assume
$S, T \neq \emptyset$ and $N^+(T) \neq \emptyset$.
$N^+(T) \cap C = \emptyset$ and so $N^+(T) \cap S \neq \emptyset$.

Thus $\exists x \in T, y \in S$ such that $xy \in A$.

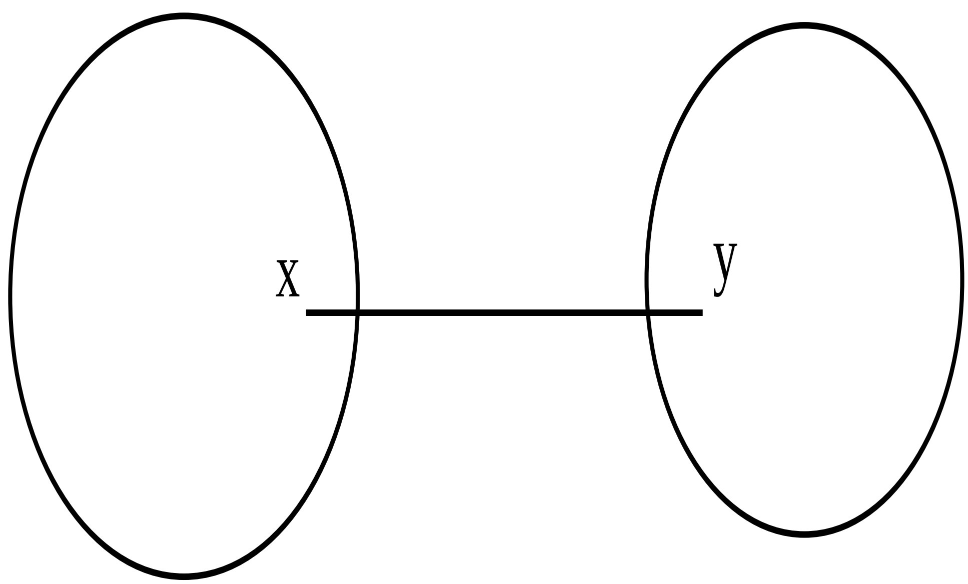


The cycle $(v_1, x, y, v_3, \ldots, v_k, v_1)$ is a cycle of length $k + 1$.

Robbin's Theorem

Theorem 5 *A connected graph G has an orientation which is strongly connected iff G is 2-edge connected.*

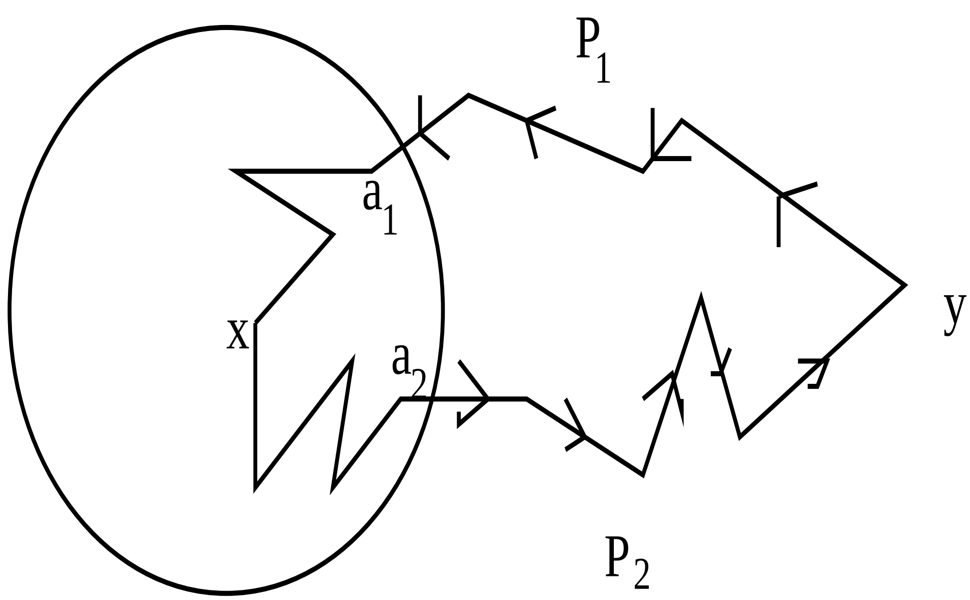
Only if: Suppose that G has a cut edge $e = xy$.



If we orient e from x to y (resp. y to x) then there is no directed path from y to x (resp. x to y).

If: Suppose G is 2-edge connected. It contains a cycle C which we can orient to produce a directed cycle.

At a general stage of the process we have a set of vertices $S \supseteq C$ and an orientation of the edges of $G[S]$ which is strongly connected.



If $S \neq V$ choose $x \in S$, $y \notin S$.
There are 2 edge disjoint paths P_1, P_2 joining y to x.
Let a_i be the first vertex of P_i which is in S.
Orient $P_1[y, a_1]$ from y to a_1.
Orient $P_2[y, a_2]$ from a_2 to y.

Claim: The subgraph $G[S \cup P_1 \cup P_2]$ is strongly connected.

Let $S' = S \cup P_1 \cup P_2$. We must show that there is a directed path from α to β for all $\alpha, \beta \in S'$.

(i) $\alpha, \beta \in S$: $\exists$ a directed path from α to β in S.
(ii) $\alpha \in S$, $\beta \in P_1 \setminus S$: Go from α to a_2 in S, from a_2 to y on P_2, from y to β along P_1.
(iii) $\alpha \in S$, $\beta \in P_2 \setminus S$: Go from α to a_2 in S, from a_2 to β on P_2.
(iv) $\alpha \in P_1 \setminus S$, $\beta \in S$: Go from α to α_1 on P_1, from a_1 to β in S.
(v) $\alpha \in P_2 \setminus S$, $\beta \in S$: Go from α to y on P_1, from y to a_1 on P_1, from a_1 to β in S.

Continuing in this way we can orient the whole graph.

$\square$

Directed Euler Tours

An Euler tour of a digraph D is a directed walk which traverses each arc of D exactly once.

Theorem 6 *A digraph D has an Euler tour iff $G(D)$ is connected and $d^+(v) = d^-(v)$ for all $v \in V$.*

Proof This is similar to the undirected case.
If: Suppose $W = (v_1, v_2, \ldots, v_m, v_1)$ ($m = |A|$) is an Euler Tour. Fix $v \in V$. Whenever W visits v it enters through a new arc and leaves through a new arc. Thus each visit requires one entering arc and one leaving arc. Thus $d^+(v) = d^-(v)$.

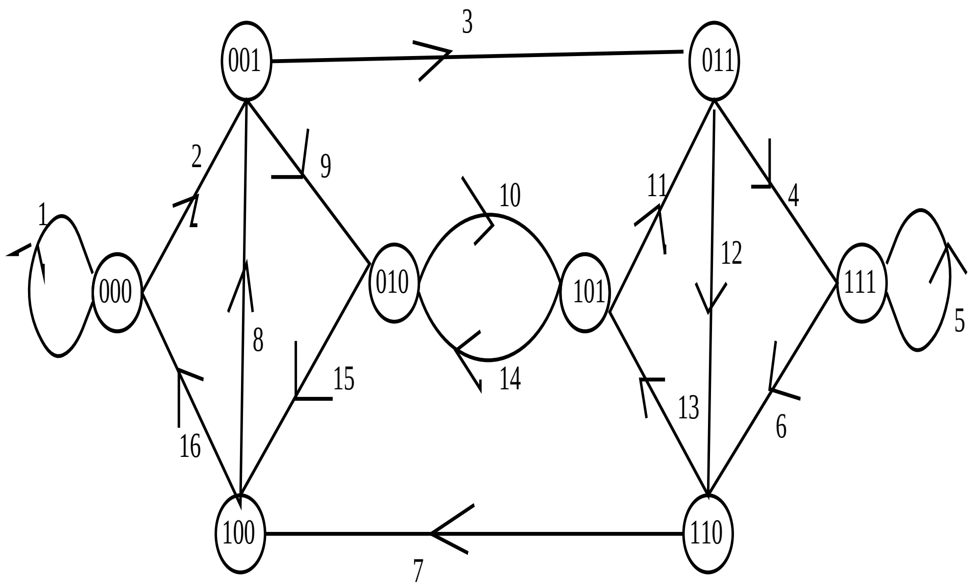
Only if: We use induction on the number of arcs. D is not a DAG as it has no sources or sinks. Thus it must have a directed cycle C. Now remove the edges of C. Each component C_i of $G(D-C)$ satisfies the degree conditions and so contains an Euler tour W_i. Now, as in the undirected case, go round the cycle C and the first time you vist C_i add the tour W_i. This produces an Euler tour of the whole digraph D.

As a simple application of the previous theorem we consider the following problem. A 0-1 sequence $x = (x_1, x_2, \ldots, x_m)$ has proprty P_n if for every 0-1 sequence $y = (y_1, y_2, \ldots, y_n$ there is an index k such that $x_k = y_1, x_{k+1} = y_2, \ldots, x_{k+n-1} = y_n$. Here $x_t = x_{m+1-t}$ if $t > m$.

Note that we must have $m \geq 2^n$ in order to have a distinct k for each possible x.

Theorem 7 *There exists a sequence of length 2^n with property P_n.*

Proof Define the digraph D_n with vertex set $\{0, 1\}^{n-1}$ and 2^n directed arcs of the form $((p_1, p_2, \ldots, p_{n-1}), (p_2, p_3, \ldots, p_n))$.
$G(D_n)$ is connected as we can join $(p_1, p_2, \ldots, p_{n-1})$ to $(q_1, q_2, \ldots, q_{n-1})$ by the path $(p_1, p_2, \ldots, p_{n-1})$, $(p_2, p_3 \ldots, p_{n-1}, q_1), (p_3, p_4, \ldots, p_{n-1}, q_1, q_2), \ldots,$ $(q_1, q_2, \ldots, q_{n-1})$. Each vertex of D_n has indegree and outdegree 2 and so it has an Euler tour W.



Suppose that W visits the vertices of D_n in the sequence $(v_1, v_2, \ldots, v_{2^n}$. Let x_i be the first bit of v_i. We claim that $x_1, x_2, \ldots, x_{2^n}$ has property P_n. Give arc $((p_1, p_2, \ldots, p_{n-1}), (p_2, p_3, \ldots, p_n))$ the label $(p_1, p_2, \ldots, p_n)$. No other arc has this label.

Given $(y_1, y_2, \ldots, y_n)$ let k be such that (v_k, v_{k+1}) has this label. Then $v_k = (y_1, y_2, \ldots, y_{n-1})$ and $v_{k+1} = (y_2, y_3, \ldots, y_n)$ and then $x_k = y_1, x_{k+1} = y_2, \ldots, x_{k+n-1} = y_n$. $\square$