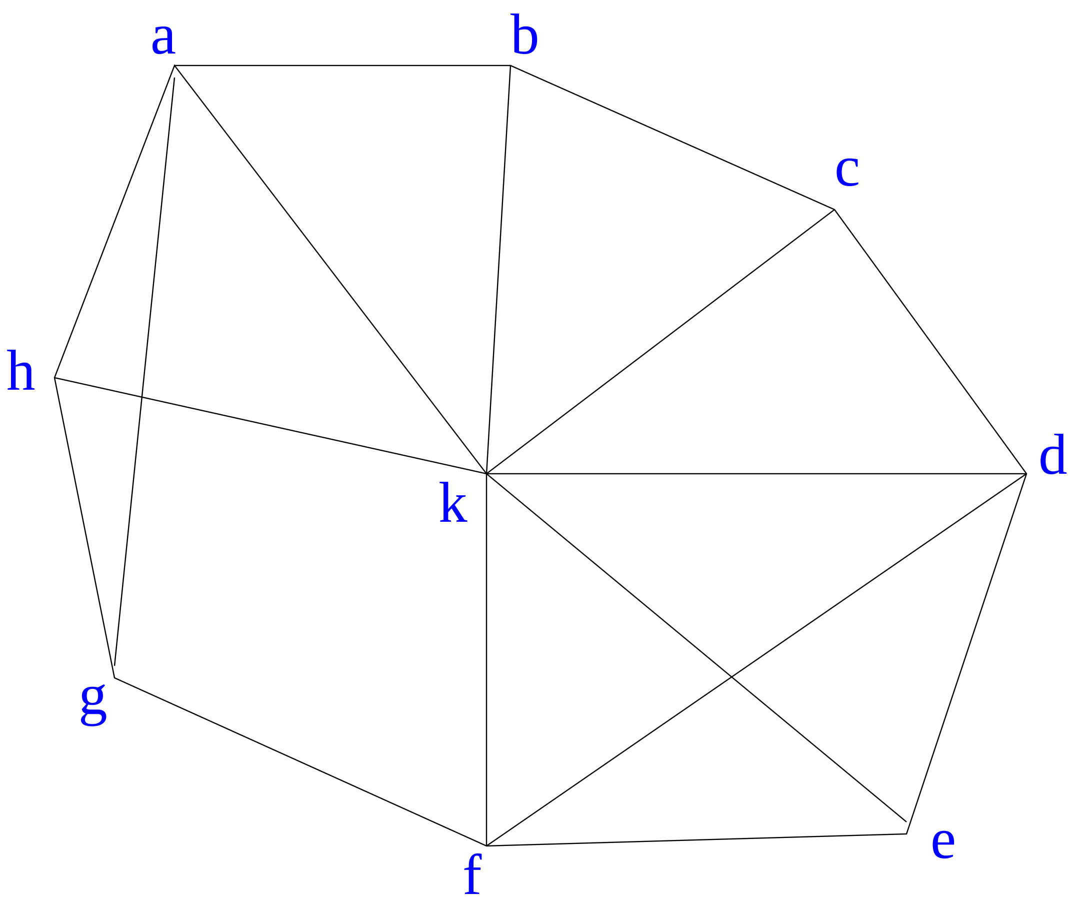
Graph Theory

Simple Graph $G = (V, E)$.
V={vertices}, E={edges}.



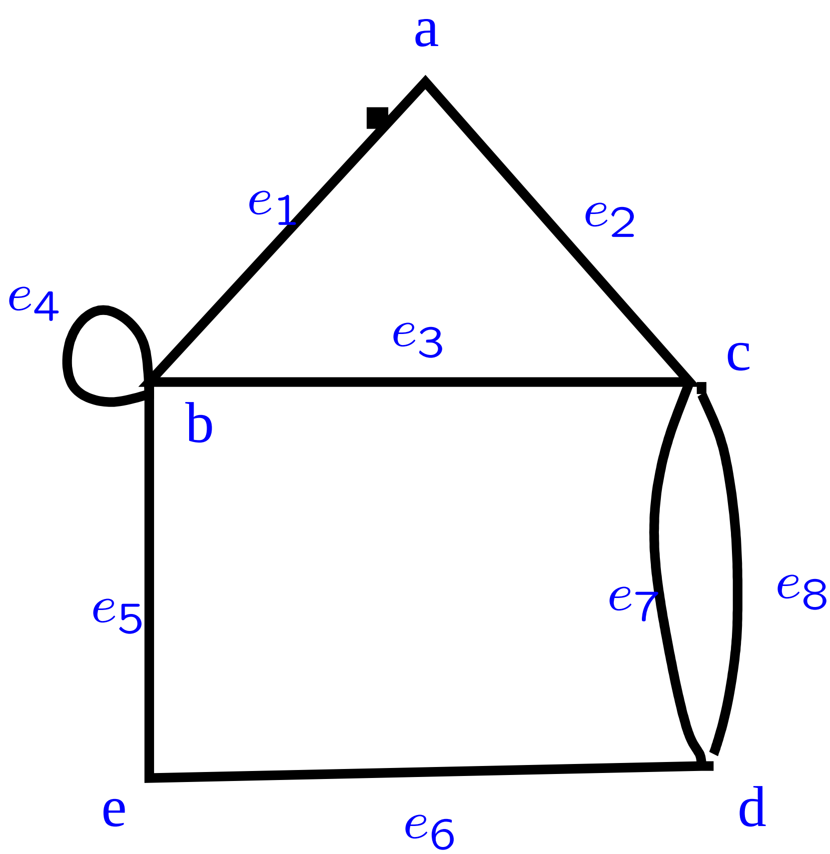
V={a,b,c,d,e,f,g,h,k}

E={(a,b),(a,g),(a,h),(a,k),(b,c),(b,k),...,(h,k)} |E|=16.

Graph or Multi-Graph

We allow loops and multiple edges.

$G = (V, E.\psi)$

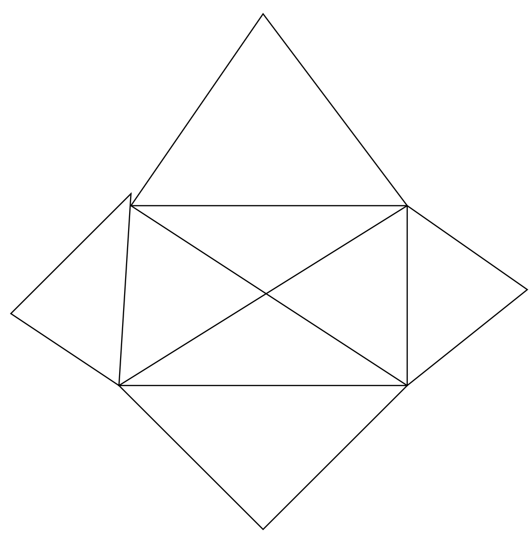


$$V = \{a, b, c, d, e\},\ E = \{e_1, e_2, \ldots, e_8\}.$$

t	1	2	3	4	5	6	7	8
ψ(t)	ab	ae	be	bb	bc	cd	de	de

Eulerian Graphs

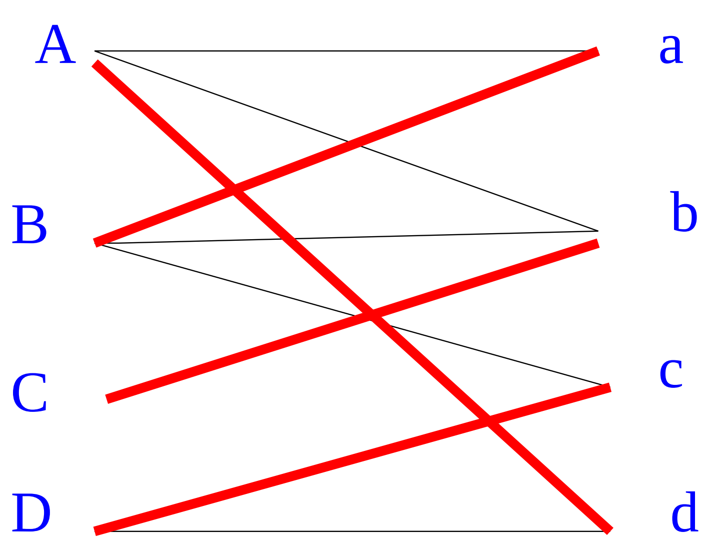
Can you draw the diagram below without taking your pen off the paper or going over the same line twice?



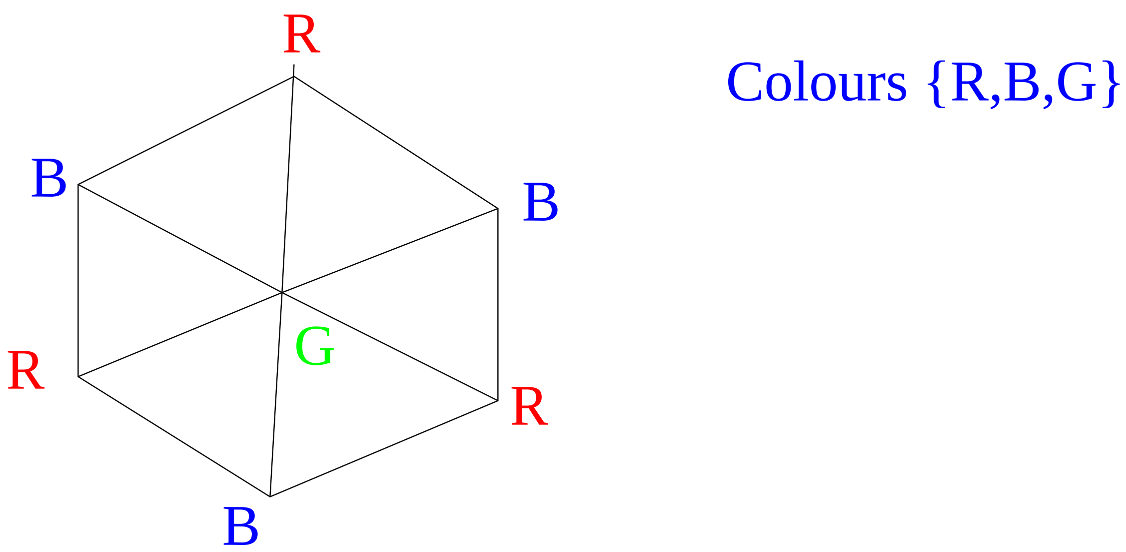
Bipartite Graphs

G is bipartite if $V = X \cup Y$ where X and Y are disjoint and every edge is of the form (x, y) where $x \in X$ and $y \in Y$.

In the diagram below, A,B,C,D are women and a,b,c,d are men. T here is an edge joining x and y iff x and y like each other. The red edges form a "perfect matching" enabling everybody to be paired with someone they like. Not all graphs will have perfect matching!



Vertex Colouring



Let $C = \{colours\}$. A vertex colouring of G is a map $f : V \rightarrow C$. We say that $v \in V$ gets coloured with $f(v)$.

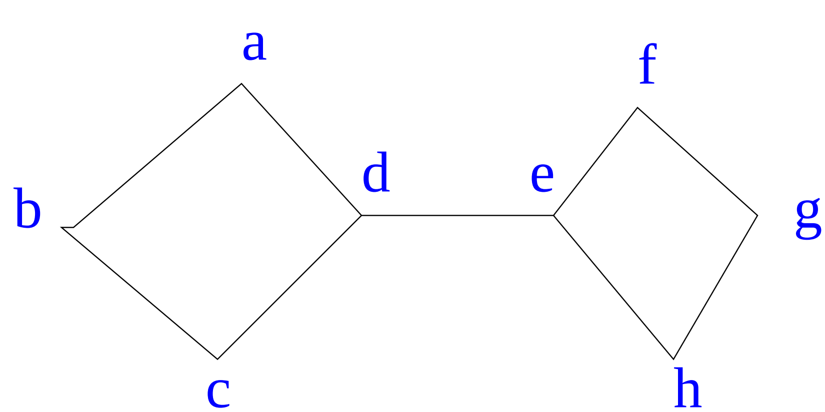
The colouring is proper iff $(a, b) \in E \Rightarrow f(a) \neq f(b)$.

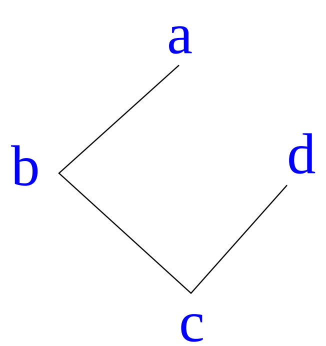
The *Chromatic Number* $\chi(G)$ is the minimum number of colours in a proper colouring.

Application: V={exams}. (a, b) is an edge iff there is some student who needs to take both exams. $\chi(G)$ is the minimum number of periods required in order that no student is scheduled to take two exams at once.

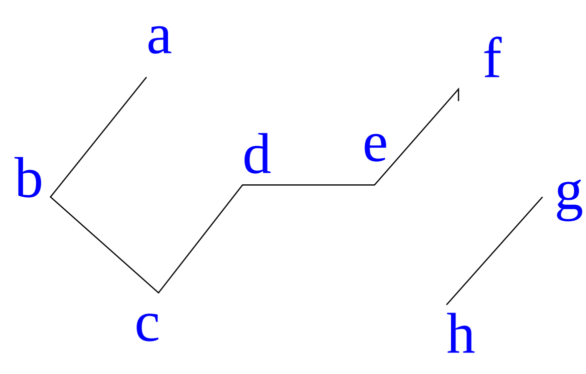
Subgraphs

$G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
G' is a *spanning* subgraph if $V' = V$.





NOT SPANNING

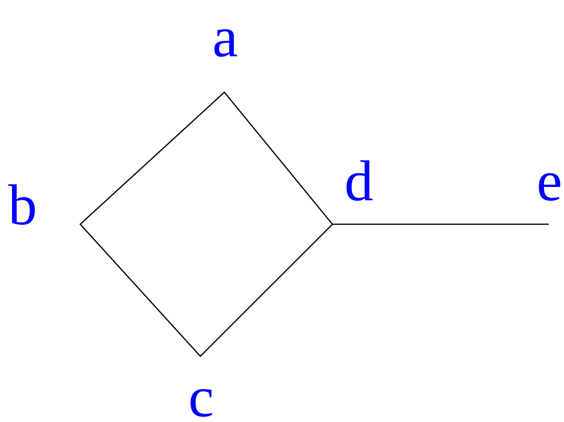


SPANNING

If $V' \subseteq V$ then

$$G[V'] = (V', \{(u,v) \in E : \ u,v \in V'\})$$

is the subgraph of G induced by V'.



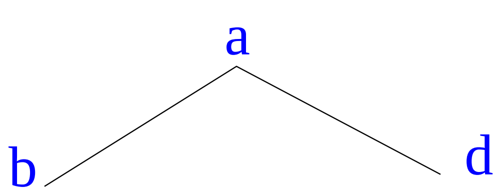
G[{a,b,c,d,e}]

Similarly, if $E_1 \subseteq E$ then $G[E_1] = (V_1, E_1)$ where

$$V_1 = \{v \in V_1 : \exists e \in E_1 \text{ such that } v \in e\}$$

is also induced (by E_1).

E_1 = {(a,b), (a,d)}

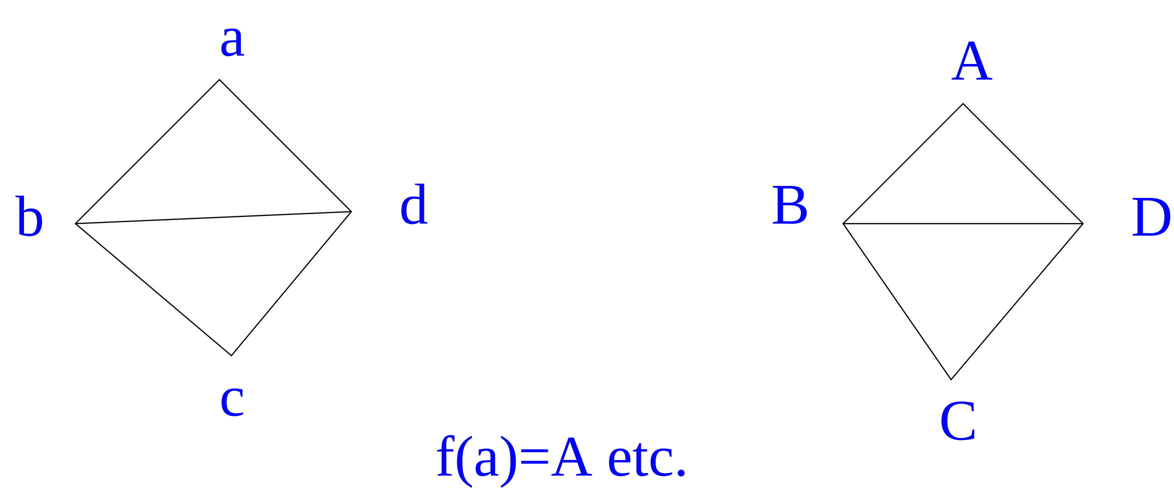


G[E_1]

Isomorphism for Simple Graphs

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that

$$(v, w) \in E_1 \leftrightarrow (f(v), f(w)) \in E_2.$$



Isomorphism for Graphs

$G_1 = (V_1, E_1, \psi_1)$ and $G_2 = (V_2, E_2, \psi_2)$ are *isomorphic* if there exist bijections $f : V_1 \to V_2$ and $g : E_1 \to E_2$ such that

$$\psi_1(e) = ab \leftrightarrow \psi_2(g(e)) = f(a)f(b).$$

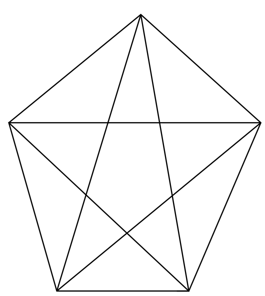
Complete Graphs

$$K_n = ([n], \{(i,j) :\ 1 \le i < j \le n\})$$

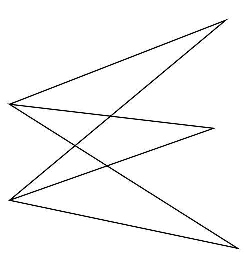
is the complete graph on n vertices.

$$K_{m,n} = ([m] \cup [n],\ \{(i,j) :\ i \in [m], j \in [n]\})$$

is the complete bipartite graph on $m + n$ vertices.
(The notation is a little imprecise but hopefully clear.)



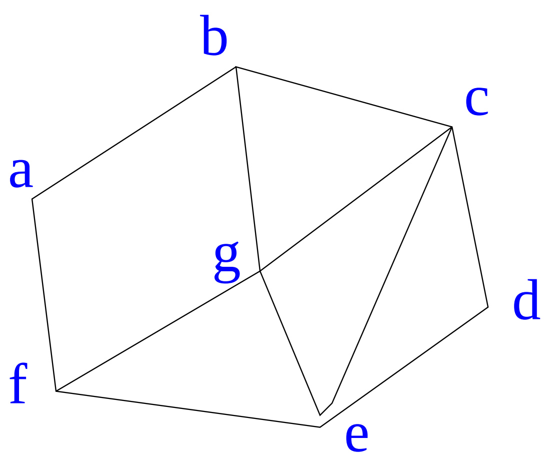
K_5



$K_{2,3}$

Vertex Degrees

$$
\begin{aligned}
d_G(v) &= \text{degree of vertex } v \text{ in } G \\
&= \text{number of edges incident with } v \\
\delta(G) &= \min_v d_G(v) \\
\Delta(G) &= \max_v d_G(v)
\end{aligned}
$$



G

$d_G(a) = 2, d_G(g) = 4$ etc.

$\delta(G) = 2, \Delta(G) = 4$

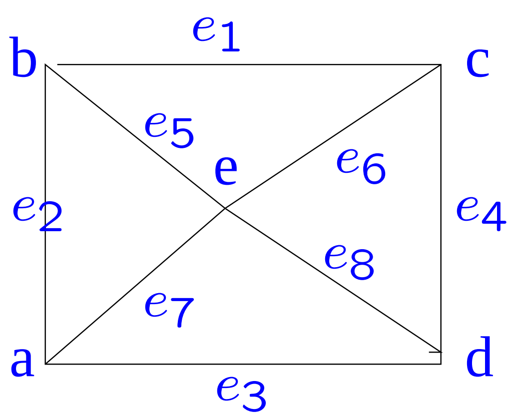
If $V = \{1, 2, \ldots, n\}$ then $\mathbf{d} = d_1, d_2, \ldots, d_n$ where $d_j = d_G(j)$ is called the degree sequence of G.

Matrices and Graphs

Incidence matrix M: $V \times E$ matrix.

$$M(v,e) = \begin{cases} 1 & v \in e \\ 0 & v \notin e \end{cases}$$

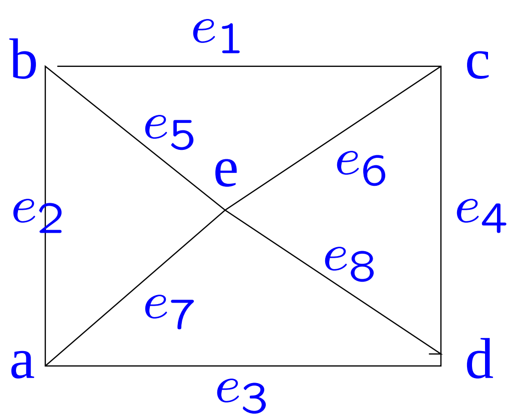
	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a		1	1				1	
b	1	1			1			
c	1			1		1		
d			1	1				1
e					1	1	1	1



Adjacency matrix A: $V \times V$ matrix.

$$A(v, w) = \text{ number of } v, w \text{ edges.}$$

	a	b	c	d	e
a		1		1	1
b	1		1		1
c		1		1	1
d	1		1		1
e	1	1	1	1	



Theorem 1

$$\sum_{v \in V} d_G(v) = 2|E|$$

Proof Consider the incidence matrix M. Row v has $d_G(v)$ 1's. So

$$\# \text{ 1's in matrix } M \text{ is } \sum_{v \in V} d_G(v).$$

Column e has two 1's. So

$$\# \text{ 1's in matrix } M \text{ is } 2|E|.$$

$\square$

Corollary 1 *In any graph, the number of vertices of odd degree, is even.*

Proof Let ODD = $\{$odd degree vertices$\}$ and $EVEN = V \setminus ODD$.

$$\sum_{v \in ODD} d(v) = 2|E| - \sum_{v \in EVEN} d(v)$$

is even.

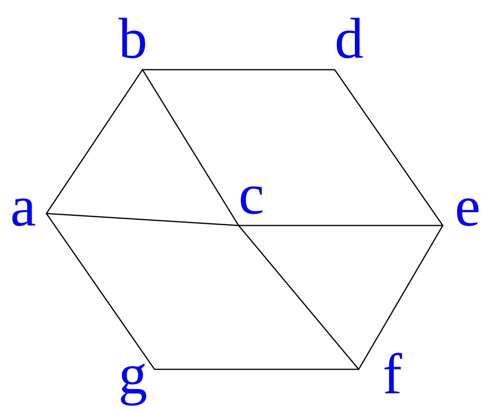
So $|ODD|$ is even. $\square$

Paths and Walks

$W = (v_1, v_2, \ldots, v_k)$ is a walk in G if $(v_i, v_{i+1}) \in E$ for $1 \leq i < k$.

A path is a walk in which the vertices are distinct.

W_1 is a path, but W_2, W_3 are not.



W_1 = a,b,c,e,d

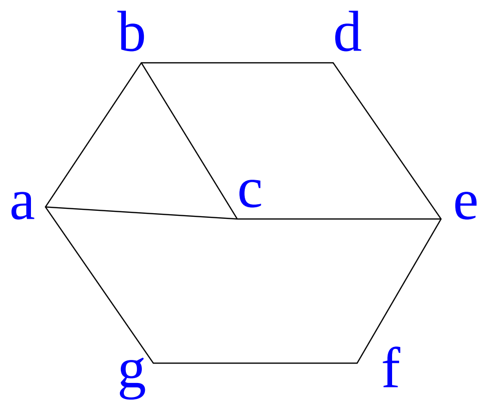
W_2 =a,b,a,c,e

W_3 =g,f,c,e,f

A walk is closed if $v_1 = v_k$. A cycle is a closed walk in which the vertices are distinct except for v_1, v_k.

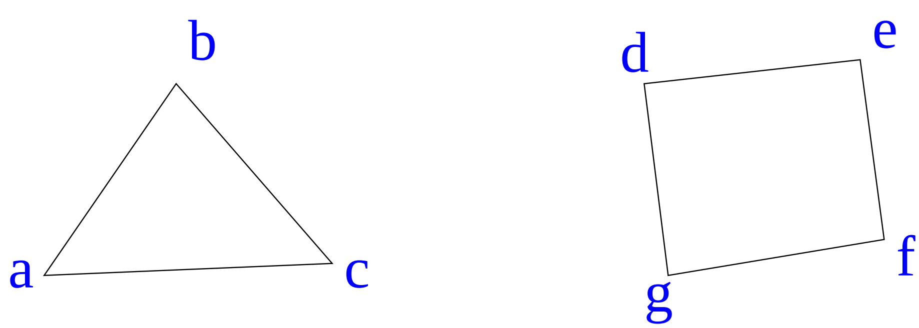
b, c, e, d, b is a cycle.

b, c, a, b, d, e, c, b is not a cycle.



Connected components

We define a relation $\sim$ on V.
$a \sim b$ iff there is a walk from a to b.



$a \sim b$ but $a \not\sim d$.

Claim: $\sim$ is an equivalence relation.

Reflexivity $v \sim v$ as v is a (trivial) walk from v to v.

Symmetry $u \sim v$ implies $v \sim u$.
$(u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v implies $(u_k, u_{k-1}, \ldots, u_1)$ is a walk from v to u.

Transitivity $u \sim v$ and $v \sim w$ implies $u \sim w$.

$W_1 = (u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v and $W_2 = (v_1 = v, v_2, v_3, \ldots, v_\ell = w)$ is a walk from v to w imples that $(W_1, W_2) = (u_1, u_2 \ldots, u_k, v_2, v_3, \ldots, v_\ell)$ is a walk from u to w.

The equivalence classes of $\sim$ are called *connected components*.

In general $V = C_1 \cup V_2 \cup \cdots \cup C_r$ where $C_1, C_2, \ldots,$ C_r are the connected comonents.

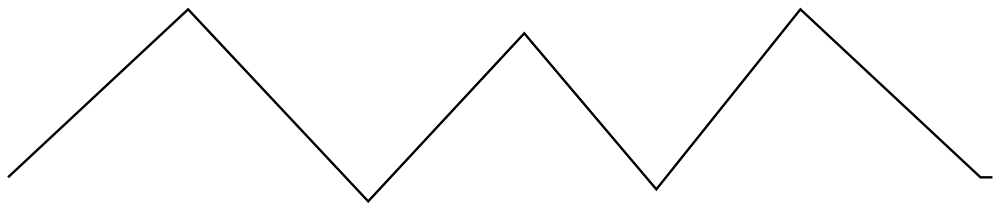
We let $\omega(G)(= r)$ be the number of components of G.

G is *connected* iff $\omega(G) = 1$ i.e. there is a walk between every pair of vertices.

Thus $C_1, C_2, \ldots, C_r$ induce connected subgraphs $G[C_1], \ldots, G[C_r]$ of G

Paths and walks

For a walk W we let $\ell(W)$ = no. of edges in W.



$\ell(W) = 6$

Lemma 1 *Suppose W is a walk from vertex a to vertex b and that W minimises ℓ over all walks from a to b. Then W is a path.*

Proof Suppose $W = (a = a_0, a_1, \ldots, a_k = b)$ and $a_i = a_j$ where $0 \leq i < j \leq k$. Then $W' = (a_0, a_1, \ldots, a_i, a_{j+1}, \ldots, a_k)$ is also a walk from a to b and $\ell(W') = \ell(W) - (j - i) < \ell(W)$ – contradiction. □

Corollary 2 *If $a \sim b$ then there is a path from a to b.*

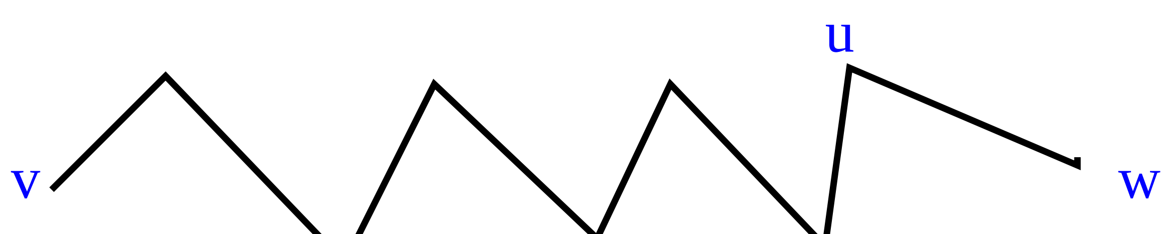
So G is connected $\leftrightarrow \forall a, b \in V$ there is a path from a to b.

Walks and powers of matrices

Theorem 2 $A^k(v,w)$ = *number of walks of length* k *from* v *to* w *with* k *edges.*

Proof By induction on k. Trivially true for $k = 1$. Assume true for some $k \geq 1$.

Let $N_t(v,w)$ be the number of walks from v to w with t edges.
Let $N_t(v,w;u)$ be the number of walks from v to w with t edges whose penultimate vertex is u.



$$
\begin{aligned}
N_{k+1}(v,w) &= \sum_{u \in V} N_{k+1}(v,w;u) \\
&= \sum_{u \in V} N_k(v,u) A(u,w) \\
&= \sum_{u \in V} A^k(v,u) A(u,w) && \text{induction} \\
&= A^{k+1}(v,w).
\end{aligned}
$$

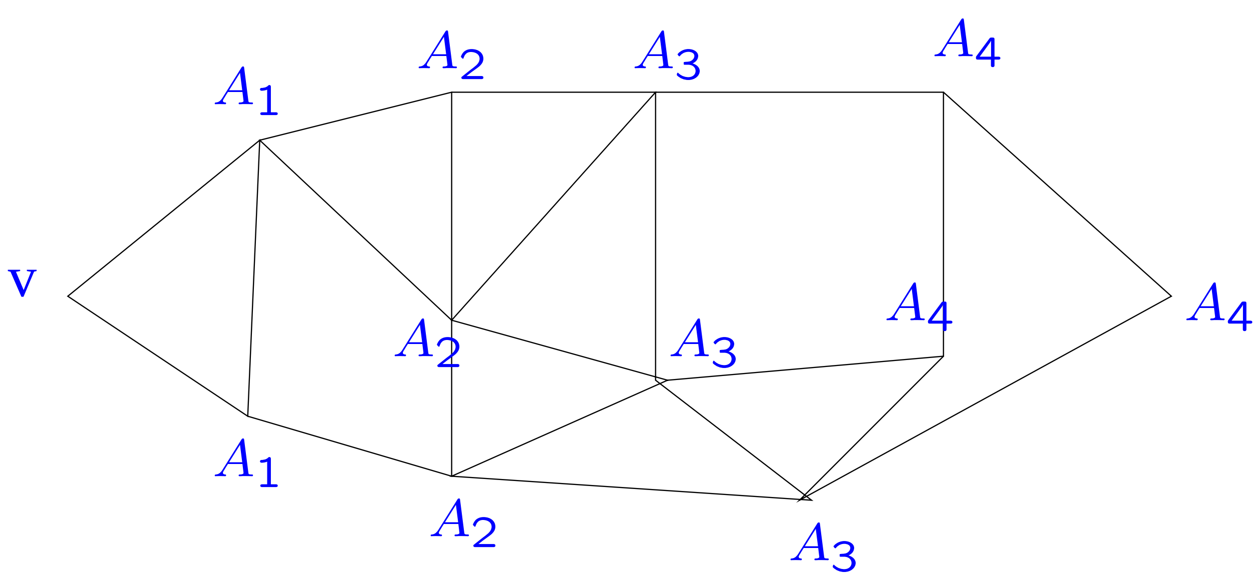
Breadth First Search – BFS

Fix $v \in V$. For $w \in V$ let

$d(v,w) =$ minimum number of edges in a path from v to w.

For $t = 0, 1, 2, \ldots,$ let

$$A_t = \{w \in V :\ d(v,w) = t\}.$$



$A_0 = \{v\}$ and $v \sim w \leftrightarrow d(v,w) < \infty$.

In BFS we construct $A_0, A_1, A_2, \ldots,$ by

$$A_{t+1} = \{w \notin A_0 \cup A_1 \cup \cdots \cup A_t : \exists \text{ an edge } (u, w) \text{ such that } u \in A_t\}.$$

$$\text{Note}: \quad \text{no edges } (a, b) \text{ between } A_k \text{ and } A_\ell \text{ for } \ell - k \geq 2, \text{ else } w \in A_{k+1} \neq A_\ell. \tag{1}$$

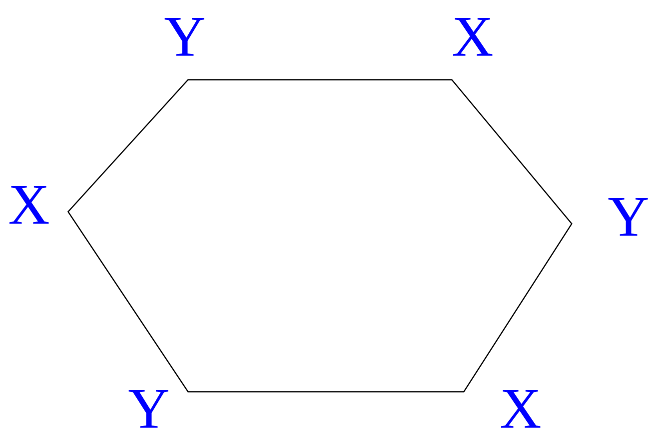
In this way we can find all vertices in the same component C as v.

By repeating for $v' \notin C$ we find another component etc.

Characterisation of bipartite graphs

Theorem 3 *G is bipartite $\leftrightarrow$ G has no cycles of odd length.*

Proof $\rightarrow$: $G = (X \cup Y, E)$.



Typical Cycle

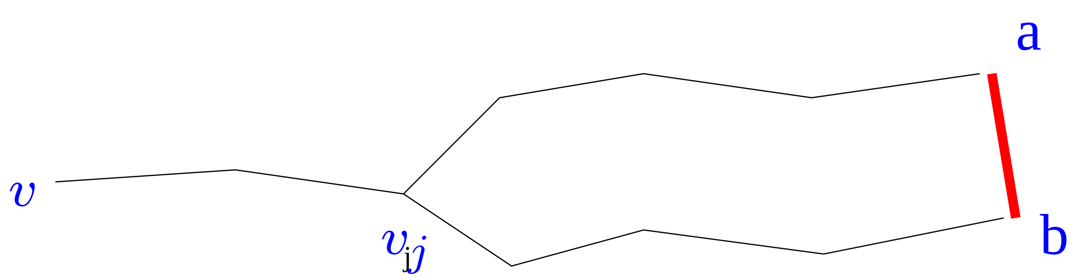
Suppose $C = (u_1, u_2, \ldots, u_k, u_1)$ is a cycle. Suppose $u_1 \in X$. Then $u_2 \in Y, u_3 \in X, \ldots, u_k \in Y$ implies k is even.

$\leftarrow$ Assume G is connected, else apply following argument to each component.
Choose $v \in V$ and construct $A_0, A_1, A_2, \ldots,$ by BFS.

$X = A_0 \cup A_2 \cup A_4 \cup \cdots$ and $Y = A_1 \cup A_3 \cup A_5 \cup \cdots$

We need only show that X and Y contain no edges and then all edges must join X and Y. Suppose X contains edge (a, b) where $a \in A_k$ and $b \in A_\ell$.

(i) If $k \neq \ell$ then $|k - \ell| \geq 2$ which contradicts (1)
(ii) $k = \ell$:



There exist paths $(v = v_0, v_1, v_2, \ldots, v_k = a)$ and $(v = w_0, w_1, w_2, \ldots, w_k = b)$.

Let $j = \max\{t : \ v_t = w_t\}$.

$$(v_j, v_{j+1}, \ldots, v_k, w_k, w_{k-1}, \ldots, w_j)$$

is an odd cycle – length $2(k - j) + 1$ – contradiction.

$\square$