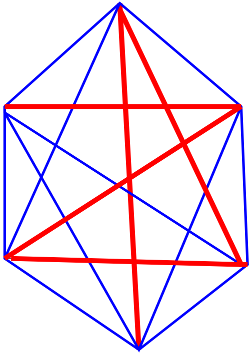
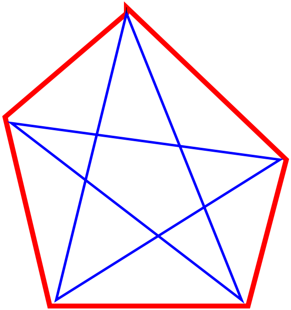
RAMSEY THEORY

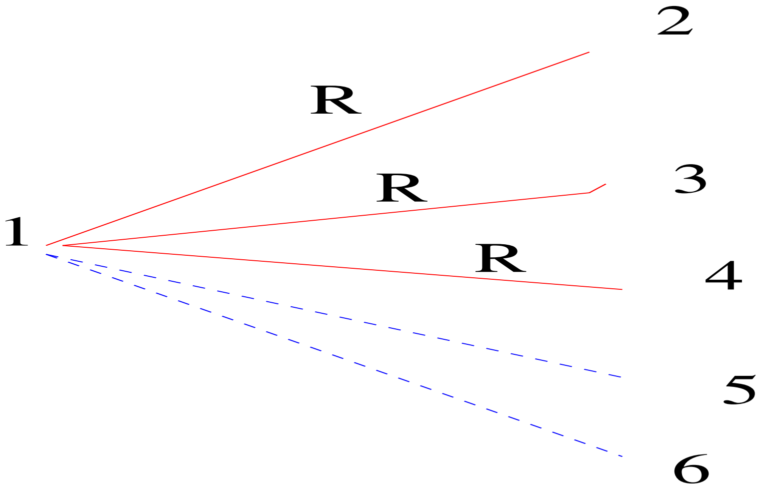
Ramsey's Theorem

Suppose we 2-colour the edges of K_6 of Red and Blue. There *must* be either a Red triangle or a Blue triangle.





This is not true for K_5.



There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter implies (1,2,3) is a Red triangle.

Ramsey's Theorem

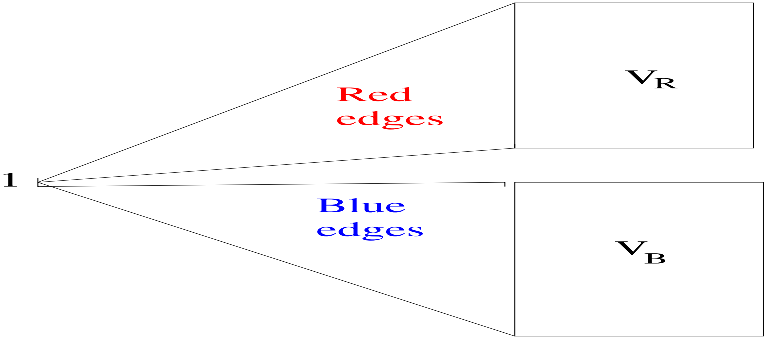
For all positive integers k, ℓ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of K_N are coloured Red or Blue then then either there is a "Red k-clique" or there is a "Blue ℓ-clique. A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

$$\begin{aligned} R(1, k) &= R(k, 1) &= 1 \\ R(2, k) &= R(k, 2) &= k \end{aligned}$$

Theorem

$$R(k,\ell) \le R(k,\ell-1) + R(k-1,\ell).$$

Proof Let $N = R(k,\ell-1) + R(k-1,\ell)$.



$V_R = \{(x : (1,x)$ is coloured Red$\}$ and $V_B = \{(x : (1,x)$ is coloured Blue$\}$.

$$|V_R| \geq R(k-1,\ell) \text{ or } |V_B| \geq R(k,\ell-1).$$

Since

$$\begin{aligned} |V_R| + |V_B| &= N-1 \\ &= R(k,\ell-1) + R(k-1,\ell) - 1. \end{aligned}$$

Suppose for example that $|V_R| \geq R(k-1,\ell)$. Then either V_R contains a Blue ℓ-clique – done, or it contains a Red $k-1$-clique K. But then $K \cup \{1\}$ is a Red k-clique. Similarly, if $|V_B| \geq R(k,\ell-1)$ then either V_B contains a Red k-clique – done, or it contains a Blue $\ell-1$-clique L and then $L \cup \{1\}$ is a Blue ℓ-clique. □

Theorem

$$R(k,\ell) \le \binom{k+\ell-2}{k-1}.$$

Proof Induction on $k+\ell$. True for $k+\ell \le 5$ say. Then

$$\begin{aligned} R(k,\ell) &\le R(k,\ell-1) + R(k-1,\ell) \\ &\le \binom{k+\ell-3}{k-1} + \binom{k+\ell-3}{k-2} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

□

So, for example,

$$\begin{aligned} R(k,k) &\le \binom{2k-2}{k-1} \\ &\le 4^k \end{aligned}$$

Theorem

$$R(k,k) > 2^{k/2}$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.
We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: {There is a Red k-clique} and
$\mathcal{E}_B$ be the event: {There is a Blue k-clique}.
We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.

Let $\mathcal{E}_{R,j}$ be the event: {C_j is Red} and let $\mathcal{E}_{B,j}$ be the event: {C_j is Blue}.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\le \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) = 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \le 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\le 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\le 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
$$

Very few of the Ramsey numbers are known exactly. Here are a few known values.

$$
\begin{aligned}
R(3,3) &= 6 \\
R(3,4) &= 9 \\
R(4,4) &= 18 \\
R(4,5) &= 25 \\
43 \leq R(5,5) &\leq 49
\end{aligned}
$$

Ramsey's Theorem in general

Remember that the elements of $\binom{S}{r}$ are the r-subsets of S

Theorem

Let $r, s \geq 1$, $q_i \geq r$, $1 \leq i \leq s$ be given. Then there exists $N = N(q_1, q_2, \ldots, q_s; r)$ with the following property: Suppose that S is a set with $n \geq N$ elements. Let each of the elements of $\binom{S}{r}$ be given one of s colors. .

Then there exists i and a q_i-subset T of S such that all of the elements of $\binom{T}{r}$ are colored with the ith color.

Proof First assume that $s = 2$ i.e. two colors, Red, Blue.

Ramsey's Theorem in general

Note that

$$\begin{aligned}(a)\ N(p,q;1) &= p+q-1\\ (b)\ N(p,r;r) &= p(\geq r)\\ N(r,q;r) &= q(\geq r)\end{aligned}$$

We proceed by induction on r. It is true for $r = 1$ and so assume $r \geq 2$ and it is true for $r - 1$ and arbitrary p, q. Now we further proceed by induction on $p + q$. It is true for $p + q = 2r$ and so assume it is true for r and all p', q' with $p' + q' < p + q$.
Let

$$\begin{aligned}p_1 &= N(p-1,q;r)\\ p_2 &= N(p,q-1;r)\end{aligned}$$

These exist by induction.

Ramsey's Theorem in general

Now we prove that

$$N(p, q; r) \le 1 + N(p_1, q_1; r - 1)$$

where the RHS exists by induction.

Suppose that $n \ge 1 + N(p_1, q_1; r - 1)$ and we color $\binom{[n]}{r}$ with 2 colors. Call this coloring σ.

From this we define a coloring τ of $\binom{[n-1]}{r-1}$ as follows: If $X \in \binom{[n-1]}{r-1}$ then give it the color of $X \cup \{n\}$ under σ.

Now either (i) there exists $A \subseteq [n-1]$, $|A| = p_1$ such that (under τ) all members of $\binom{A}{r-1}$ are Red or (ii) there exists $B \subseteq [n-1]$, $|A| = q_1$ such that (under τ) all members of $\binom{B}{r-1}$ are Blue.

Ramsey's Theorem in general

Assume w.l.o.g. that (i) holds.

$$|A| = p_1 = N(p-1, q; r).$$

Then either

(a) $\exists B \subseteq A$ such that $|B| = q$ and under σ all of $\binom{B}{r}$ is Blue,

or

(b) $\exists A' \subseteq A$ such that $|A'| = p - 1$ and all of $\binom{A'}{r}$ is Red. But then all of $\binom{A' \cup \{n\}}{r}$ is Red. If $X \subseteq A'$, $|X| = r - 1$ then τ colors X Red, since $A' \subseteq A$. But then σ will color $X \cup \{n\}$ Red.

Ramsey's Theorem in general

Now consider the case of s colors. We show that

$$N(q_1, q_2, \ldots, q_s; r) \leq N(Q_1, Q_2; r)$$

where

$$\begin{aligned} Q_1 &= N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r) \\ Q_2 &= N(q_{\lfloor s/2 \rfloor + 1}, q_{\lfloor s/2 \rfloor + 2}, \ldots, q_s; r) \end{aligned}$$

Let $n = N(Q_1, Q_2; r)$ and assume we are given an s-coloring of $\binom{[n]}{r}$.

First temporarily re-color Red, any r-set colored with $i \leq \lfloor s/2 \rfloor$ and re-color Blue any r-set colored with $i > \lfloor s/2 \rfloor$.

Ramsey's Theorem in general

Then either (a) there exists a Q_1-subset A of $[n]$ with $\binom{A}{r}$ colored Red or (b) there exists a Q_2-subset B of $[n]$ with $\binom{A}{r}$ colored Blue.

W.l.o.g. assume the first case. Now replace the colors of the r-sets of A by there original colors. We have a $\lfloor s/2 \rfloor$-coloring of $\binom{A}{r}$. Since $|A| = N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r)$ there must exist some $i \leq \lfloor s/2 \rfloor$ and a q_i-subset S of A such that all of $\binom{S}{r}$ has color i.

□

Hales-Jewett Theorem

The n-dimensional m-cube is defined as

$$[m]^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : \ 1 \leq x_i \leq m\}.$$

A subset $L \subseteq [m]^n$ is a combinatorial line if there exists a non-empty set $I \subseteq [n]$ and integers $a_i, i \notin I$ such that

$$L = \{\mathbf{x} \in [m]^n : x_i = a_i, i \notin I \text{ and } x_i = x_j \text{ for } i, j \in I\}.$$

For example if $n = 5, m = 4$ and $I = \{1, 3, 5\}, a_2 = 3, a_4 = 1$ then

$$\{(1, 3, 1, 1, 1), (2, 3, 2, 1, 2), (3, 3, 3, 1, 3), (4, 3, 4, 1, 4)\} \text{ is a line.}$$

Note that the active indices in I are required to increase from 1 to m.

Hales-Jewett Theorem

Theorem

For all positive integers r, m there exists a least integer $HJ(r, m)$ such that if $n \geq HJ(r, m)$ and $[m]^n$ is colored with r colors, then there is a monochromatic combinatorial line.

Proof: Note that if every r-coloring of $[m]^n$ contains a monochromatic line, then so does every r-coloring of $[m]^{n+1}$. (If we fix $x_{n+1} = 1$ then we are esentially dealing with $[m]^n$.)

Given a line L we define L^-, L^+ to be its first and last points, as the active indices increase from 1 to m.

Hales-Jewett Theorem

Lines $L_1, L_2, \ldots, L_s$ are focused at a point f if $L_i^+ = f$ for $i = 1, 2, \ldots, s$.

Lines $L_1, L_2, \ldots, L_s$ are color focused at a point f if they are focused at f and the truncated lines $L_i \setminus \{L_i^+\}$ are monochromatic with different colors.

For the proof we use induction on m. The case $m = 1$ is trivial.

We will show by induction on s that for each $1 \leq s \leq r$ there exists $N = FHJ(r, s, m)$ such that any r-coloring of $[m]^N$ contains either

1. A monochromatic line, or
2. s color-focused lines.

Hales-Jewett Theorem

The case $s = r$ implies the theorem, since at least one of the foci f_i of the r lines has the same color as the common color of the truncated lines and extend it to a whole line.

For $s = 1$ we take $FHJ(r, 1, m) = HJ(r, m-1)$, which exists by induction. (We just need to add an mth point to the monochromatic line of length $m - 1$.)

Assuming that $n \overset{d}{=} FHJ(r, s-1, m)$ exists, we claim that

$$FHJ(r, s, m) \leq N = n + n',$$

where $n' \overset{d}{=} HJ(r^{m^n}, m-1)$. (Note that the inductive assumption asssumes that $HJ(\rho, m-1)$ exists for all choices of ρ.)

An r-coloring χ of $[m]^N = [m]^n \times [m]^{n'}$ can thought of as an r^{m^n} coloring χ' of $[m]^{n'}$ where $b \in [m]^{n'}$ is “colored” with the χ-colored cube $\{(a, b) : a \in [m]^n\}$.

By induction on m, there is a line L in $[m]^{n'}$ with active coordinate set I such that the truncated line $L \setminus \{L^+\}$ is monochromatic. (This being a line in $[m-1]^{n'}$.)

In terms of χ this means that for all $a \in [m]^n$ and all $b, b' \in L \setminus \{L^+\}$, we have

$$\chi((a, b)) = \chi((a, b')) \stackrel{d}{=} \chi''(a).$$

Hales-Jewett Theorem

Now we examine χ''. By hypothesis, (induction on s), either (i) there is a monochromatic line or (ii) we can find $s-1$ color focused lines $L_1,\ldots,L_{s-1}$ in $[m]^n$ with active coordinate sets $I_1,\ldots,I_{s-1}$ and focus f. Assume that (ii) holds, implying that f has a different color to $L_i, i<s$.

For $1\leq i\leq s-1$, define L_i' to be the line in $[m]^N$ with first point (L_i^-,L^-) and active coordinates $I_i\cup I$,

Define L_s' to be the line in $[m]^N$ with first point (f,L^-) and active coordinate set I.

The lines $L_i', 1\leq i\leq s$ form a set of s color-focused lines with focus (f,L^+), completing the induction on s.
(Note that L_s' has a different color to $L_i', i<s$ because f has a different color to $L_i, i<s$.)

Van der Waerden's theorem

An *arithmetic progression* is a sequence of integers $a, a+d, \ldots, a+(m-1)d$ where $a, d > 0$. The length of the progression is m.

Theorem

There exists a positive integer $W = W(m,k)$ *such that if* $n \geq W$ *and the integers* $[n]$ *are colored with* k *colors then then* $[n]$ *contains a monochromatic arithmetic progression of lenth* m*.*

Proof Let $n = HJ(m,k)$. Let $n = HJ(m,k)$. Give $[mn]$ a k-coloring $c : [mn] \to [k]$ and define $c' : [m]^n \to [k]$ by letting $c'(x_1, x_2, \ldots, x_n) := c(x_1 + x_2 + x_n)$, for $x_1, x_2, \ldots, x_n \in [m]$.

Since $x_1 + x_2 + \cdots + x_n \leq mn$, the function is well-defined. By the definition of n, there exists a monochromatic line in $[m]^n$. Let I be the set of active coordinates and $d = |I|$.

Van der Waerden's theorem

Let $a = \sum L^-$. Note that

$$L^- = \{(x_1, x_2, \ldots, x_n) \mid x_i = a_i \text{ for i} \in I \text{ and } x_i = 1, \text{ for } i \notin I\},$$

and the jth point

$$L^j = \{(x_1, x_2, \ldots, x_n) | x_i = a_i, \text{ for } i \in I \text{ and } x_i = j, \text{ for } i \notin I\}.$$

This means $a + (j-1)d = \sum L^j$. The line is monochromatic and so $a, a+d, a+(m-1)d$ are colored the same

Thus $W(m, k) \leq mHJ(m, k)$.

Let $r_k = N(3, 3, \ldots, 3; 2)$ be the smallest n such that if we k-color the edges of K_n then there is a mono-chromatic triangle.

Theorem

For all partitions $S_1, S_2, \ldots, S_k$ of $[r_k]$, there exist i and $x, y, z \in S_i$ such that $x + y = z$.

Proof Given a partition $S_1, S_2, \ldots, S_k$ of $[n]$ where $n \geq r_k$ we define a coloring of the edges of K_n by coloring (u, v) with color j where $|u - v| \in S_j$.

There will be a mono-chromatic triangle i.e. there exist j and $x < y < z$ such that $u = y - x$, $v = z - x$, $w = z - y \in S_j$. But $u + v = w$. □

Convex Polygons

A set of points X in the plane is in general position if no 3 points of X are collinear.

Theorem

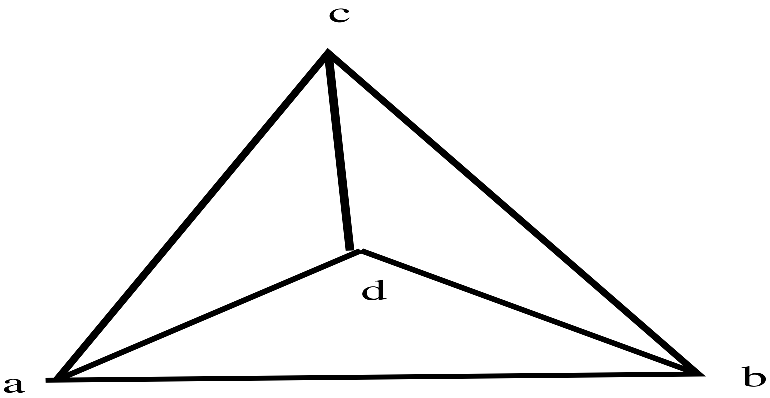
If $n \geq N(k,k;3)$ and X is a set of n points in the plane which are in general position then X contains a k-subset Y which form the vertices of a convex polygon.

Proof We first observe that if every 4-subset of $Y \subseteq X$ forms a convex quadrilateral then Y itself induces a convex polygon.

Now label the points in S from X_1 to X_n and then color each triangle $T = \{X_i, X_j, X_k\},\ i < j < k$ as follows: If traversing triangle $X_iX_jX_k$ in this order goes round it clockwise, color T Red, otherwise color T Blue.

Convex Polygons

Now there must exist a k-set T such that all triangles formed from T have the same color. All we have to show is that T does not contain the following configuration:

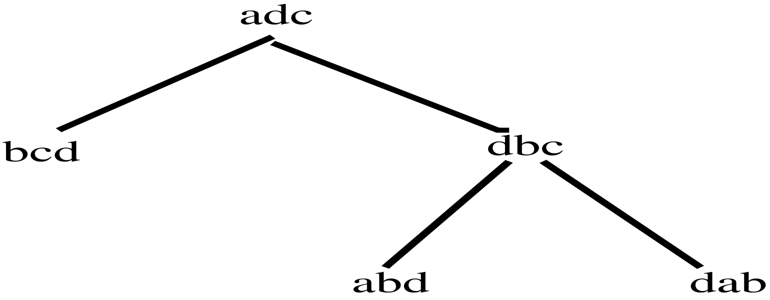


Convex Polygons

Assume w.l.o.g. that $a < b < c$ which implies that $X_i X_j X_k$ is colored Blue.

All triangles in the previous picture are colored Blue.

So the possibilities are



and all are impossible.

Ramsey $R(H_1, H_2)$

We define $r(H_1, H_2)$ to be the minimum n such that in in Red-Blue coloring of the edges of K_n there is eithere (i) a Red copy of H_1 or (ii) a Blue copy of H_2.

As an example, consider $r(P_3, P_3)$ where P_t denotes a path with t edges.

We show that

$$r(P_3, P_3) = 5.$$

$R(P_3, P_3) > 4$: We color edges incident with 1 Red and the remaining edges $\{(2,3),(3,4),(4,1)\}$ Blue. There is no mono-chromatic P_3.

$R(P_3, P_3) \leq 5$: There must be two edges of the same color incident with 1.

Assume then that $(1, 2), (1, 3)$ are both Red.

If any of $(2, 4), (2, 5), (3, 4), (3, 5)$ are Red then we have a Red P_3.

If all four of these edges are Blue then $(4, 2, 5, 3)$ is Blue.

Ramsey $R(H_1, H_2)$

We show next that $r(K_{1,s}, P_t) \leq s + t$. Here $K_{1,s}$ is a star: i.e. a vertex v and t incident edges.

Let $n = s + t$. If there is no vertex of Red degree s then the minimum degree in the graph induced by the Blue edges is at least t.

We then note that a graph of minimum degree δ contains a path of length δ.