Polya's Theory of Counting

Polya's Theory of Counting

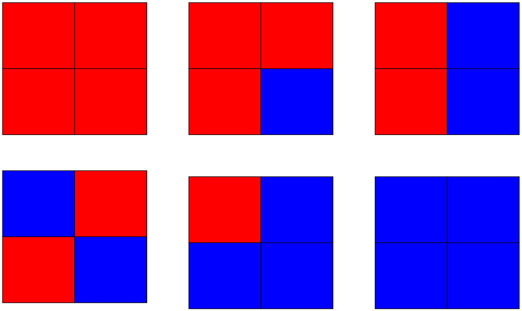
Example 1 A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into n sectors of angle $2\pi/n$. Each sector is to be colored Red or Blue. How many different colorings are there?

One could argue for 2^n.

On the other hand, what if we only distinguish colorings which cannot be obtained from one another by a rotation. For example if $n = 4$ and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.

Example 2

Now consider an $n \times n$ "chessboard" where $n \geq 2$. Here we color the squares Red and Blue and two colorings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colorings.



The general scenario that we consider is as follows: We have a set X which will stand for the set of colorings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$).

In addition there is a set G of permutations of X. This set will have a group structure:

Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that G is *closed* under compositon i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$.

We also have the following:

- A1 The *identity* permutation $1_X \in G$.
- A2 $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).
- A3 The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set G with a binary relation $\circ$ which satisfies **A1,A2,A3** is called a **Group**).

In example 1 $D = \{0, 1, 2, \ldots, n-1\}$, $X = 2^D$ and the group is $G_1 = \{e_0, e_1, \ldots, e_{n-1}\}$ where $e_j * x = x + j \; mod \; n$ stands for rotation by $2j\pi/n$.

In example 2, $X = 2^{[n]^2}$. We number the squares 1,2,3,4 in clockwise order starting at the upper left and represent X as a sequence from $\{r, b\}^4$ where for example rrbr means color 1,2,4 Red and 3 Blue. $G_2 = \{e, a, b, c, p, q, r, s\}$ is in a sense independent of n. e, a, b, c represent a rotation through $0, 90, 180, 270$ degrees respectively. p, q represent reflections in the vertical and horizontal and r, s represent reflections in the diagonals 1,3 and 2,4 respectively.

	e	a	b	c	p	q	r	s
rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr
brrr	brrr	rbrr	rrbr	rrrb	rbrr	rrrb	brrr	rrbr
rbrr	rbrr	rrbr	rrrb	brrr	brrr	rrbr	rrrb	rbrr
rrbr	rrbr	rrrb	brrr	rbrr	rrrb	rbrr	rrbr	brrr
rrrb	rrrb	brrr	rbrr	rrbr	rrbr	brrr	rbrr	rrrb
bbrr	bbrr	rbbr	rrbb	brrb	bbrr	rrbb	brrb	rbbr
rbbr	rbbr	rrbb	brrb	bbrr	brrb	rbbr	rrbb	bbrr
rrbb	rrbb	brrb	bbrr	rbbr	rrbb	bbrr	rbbr	brrb
brrb	brrb	bbrr	rbbr	rrbb	rbbr	brrb	bbrr	rrbb
rbrb	rbrb	brbr	rbrb	brbr	brbr	brbr	rbrb	rbrb
brbr	brbr	rbrb	brbr	rbrb	rbrb	rbrb	brbr	brbr
bbbr	bbbr	rbbb	brbb	bbrb	bbrb	rbbb	brbb	bbbr
bbrb	bbrb	bbbr	rbbb	brbb	bbbr	brbb	bbrb	rbbb
brbb	brbb	bbrb	bbbr	rbbb	brbb	bbrb	bbbr	brbb
rbbb	rbbb	brbb	bbrb	bbbr	brbb	bbbr	rbbb	bbrb
bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb

From now on we will write $g * x$ in place of $g(x)$.

Orbits: If $x \in X$ then its orbit

$O_x = \{y \in X : \exists g \in G \text{ such that } g * x = y\}$.

Lemma 1 The orbits partition X.

Proof $x = 1_X * x$ and so $x \in O_x$ and so $X = \bigcup_{x \in X} O_x$.

Suppose now that $O_x \cap O_y \neq \emptyset$ i.e. $\exists g_1, g_2$ such that $g_1 * x = g_2 * y$. But then for any $g \in G$ we have

$$g * x = (g \circ (g_1^{-1} \circ g_2)) * y \in O_y$$

and so $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever $O_x \cap O_y \neq \emptyset$. □

The two problems we started with are of the following form: Given a set X and a group of permutations *acting* on X, compute the number of orbits i.e. distinct colorings.

A subset H of G is called a *sub-group* of G if it satisfies *axioms* **A1,A2,A3** (with G replaced by H).

The *stabilizer* S_x of the element x is $\{g : g * x = x\}$. It is a sub-group of G.

- A1: $1_X * x = x$.
- A3: $g, h \in S_x$ implies $(g \circ h) * x = g * (h * x) = g * x = x$.

A2 holds for any subset.

Lemma 2

If $x \in X$ then $|O_x|\,|S_x| = |G|$.

Proof Fix $x \in X$ and define an equivalence relation $\sim$ on G by

$$g_1 \sim g_2 \text{ if } g_1 * x = g_2 * x.$$

Let the equivalence classes be $A_1, A_2, \ldots, A_m$. We first argue that

$$|A_i| = |S_x| \qquad i = 1, 2, \ldots, m. \tag{1}$$

Fix i and $g \in A_i$. Then

$$h \in A_i \leftrightarrow g * x = h * x \leftrightarrow (g^{-1} \circ h) * x = x$$
$$\leftrightarrow (g^{-1} \circ h) \in S_x \leftrightarrow h \in g \circ S_x$$

where $g \circ S_x = \{g \circ \sigma : \ \sigma \in S_x\}$.

Thus $|A_i| = |g \circ S_x|$. But $|g \circ S_x| = |S_x|$ since if $\sigma_1, \sigma_2 \in S_x$ and $g \circ \sigma_1 = g \circ \sigma_2$ then

$$g^{-1} \circ (g \circ \sigma_1) = (g^{-1} \circ g) \circ \sigma_1 = \sigma_1 = g^{-1} \circ (g \circ \sigma_2) = \sigma_2.$$

This proves (1).

Finally, $m = |O_x|$ since there is a distinct equivalence class for each distinct $g * x$. □

x	O_x	S_x	
rrrr	$\{rrrr\}$	G	
brrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	E
rbrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	x
rrbr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	a
rrrb	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	m
bbrr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	p
rbbr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	l
rrbb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	e
brrb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	
rbrb	{rbrb,brbr}	$\{e_0, e_2\}$	1
brbr	{rbrb,brbr}	$\{e_0, e_2\}$	
bbbr	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	$n = 4$
bbrb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
brbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
rbbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
bbbb	{bbbb}	G	

x	O_x	S_x	
rrrr	$\{e\}$	G	
brrr	{brrr,rbrr,rrbr,rrrb}	{e,r}	E
rbrr	{brrr,rbrr,rrbr,rrrb}	{e,s}	x
rrbr	{brrr,rbrr,rrbr,rrrb}	{e,r}	a
rrrb	{brrr,rbrr,rrbr,rrrb}	{e,s}	m
bbrr	{bbrr,rbbr,rrbb,brrb}	{e,p}	p
rbbr	{bbrr,rbbr,rrbb,brrb}	{e,q}	l
rrbb	{bbrr,rbbr,rrbb,brrb}	{e,p}	e
brrb	{bbrr,rbbr,rrbb,brrb}	{e,q}	
rbrb	{rbrb,brbr}	{e,b,r,s}	2
brbr	{rbrb,brbr}	{e,b,r,s}	
bbbr	{bbbr,rbbb,brbb,bbrb}	{e,s}	
bbrb	{bbbr,rbbb,brbb,bbrb}	{e,r}	
brbb	{bbbr,rbbb,brbb,bbrb}	{e,s}	
rbbb	{bbbr,rbbb,brbb,bbrb}	{e,r}	
bbbb	{e}	G	

Let $\nu_{X,G}$ denote the number of orbits.

Theorem 1

$$\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|.$$

Proof

$$\begin{aligned} \nu_{X,G} &= \sum_{x \in X} \frac{1}{|O_x|} \\ &= \sum_{x \in X} \frac{|S_x|}{|G|}, \end{aligned}$$

from Lemma 1. □

Thus in example 1 we have

$$\nu_{X,G} = \frac{1}{4}(4+1+1+1+1+1+1+1+1+2+2+1+1+1+1+4) = 6.$$

In example 2 we have

$$\nu_{X,G} = \frac{1}{8}(8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8) = 6.$$

Theorem 1 is hard to use if $|X|$ is large, even if $|G|$ is small.

For $g \in G$ let $Fix(g) = \{x \in X : \ g * x = x\}$.

Theorem 2
(Frobenius, Burnside)

$$\nu_{X,G} = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

Proof Let $A(x, g) = 1_{g*x=x}$. Then

$$\begin{aligned}
\nu_{X,G} &= \frac{1}{|G|} \sum_{x \in X} |S_x| \\
&= \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} A(x, g) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} A(x, g) \\
&= \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.
\end{aligned}$$

Let us consider example 1 with $n = 6$. We compute

g	e_0	e_1	e_2	e_3	e_4	e_5
$\|Fix(g)\|$	64	2	4	8	4	2

Applying Theorem 2 we obtain

$$\nu_{X,G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$

Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.

Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$.

Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation.

So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Example 1

First consider $e_0, e_1, \ldots, e_{n-1}$ as permutations of D.

The cycles of e_0 are $(1), (2), \ldots, (n)$.

Now suppose that $0 < m < n$. Let $a_m = gcd(m, n)$ and $k_m = n/a_m$. The cycle C_i of e_m containing the element i is is $(i, i+m, i+2m, \ldots, i+(k_m-1)m)$ since n is a divisor $k_m m$ and not a divisor of $k'm$ for $k' < k_m$. In total, the cycles of e_m are $C_0, C_1, \ldots, C_{a_m-1}$.

This is because they are disjoint and together contain n elements. (If $i + rm = i' + r'm \mod n$ then $(r-r')m + (i-i') = \ell n$. But $|i-i'| < a_m$ and so dividing by a_m we see that we must have $i = i'$.)

Next observe that if coloring x is fixed by e_m then elements on the same cycle C_i must be colored the same. Suppose for example that the color of $i + bm$ is different from the color of $i + (b + 1)m$, say Red versus Blue. Then in $e_m(x)$ the color of $i + (b + 1)m$ will be Red and so $e_m(x) \neq x$. Conversely, if elements on the same cycle of e_m have the same color then in $x \in Fix(e_m)$. This property is not peculiar to this example, as we will see.

Thus in this example we see that $|Fix(e_m)| = 2^{a_m}$ and then applying Theorem 2 we see that

$$\nu_{X,G} = \frac{1}{n} \sum_{m=0}^{n-1} 2^{gcd(m,n)}.$$

Example 2

It is straightforward to check that when n is even, we have

g	e	a	b	c	p	q	r	s
$\|Fix(g)\|$	2^{n^2}	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/2}$	$2^{n(n+1)/2}$	$2^{n(n+1)/2}$

For example, if we divide the chessboard into 4 $n/2 \times n/2$ sub-squares, numbered 1,2,3,4 then a coloring is in $Fix(a)$ iff each of these 4 sub-squares have colorings which are rotations of the coloring in square 1.

Polya's Theorem

We now extend the above analysis to answer questions like: How many *distinct* ways are there to color an 8×8 chessboard with 32 white squares and 32 black squares?
The scenario now consists of a set D (*Domain*, a set C (colors) and $X = \{x : D \to C\}$ is the set of colorings of D with the color set C. G is now a group of permutations of D.

We see first how to extend each permutation of D to a permutation of X. Suppose that $x \in X$ and $g \in G$ then we define $g * x$ by

$$g * x(d) = x(g^{-1}(d)) \qquad \text{for all } d \in D.$$

Explanation: The color of d is the color of the element $g^{-1}(d)$ which is mapped to it by g.
Consider Example 1 with $n = 4$. Suppose that $g = e_1$ i.e. rotate clockwise by $\pi/2$ and $x(1) = b, x(2) = b, x(3) = r, x(4) = r$.
Then for example

$$g * x(1) = x(g^{-1}(1)) = x(4) = r, \text{ as before.}$$

Now associate a **weight** w_c with each $c \in C$.
If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(d)}.$$

Thus, if in Example 1 we let $w(r) = R$ and $w(b) = B$ and take $x(1) = b, x(2) = b, x(3) = r, x(4) = r$ then we will write $W(x) = B^2R^2$.

For $S \subseteq X$ we define the **inventory** of S to be

$$W(S) = \sum_{x \in S} W(x).$$

The problem we discuss now is to compute the **pattern inventory** $PI = W(S^*)$ where S^* contains one member of each orbit of X under G.

For example, in the case of Example 2, with $n = 2$, we get

$$PI = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4.$$

To see that the definition of PI makes sense we need to prove

Lemma 3 If x, y are in the same orbit of X then $W(x) = W(y)$.

Proof Suppose that $g * x = y$. Then

$$\begin{aligned} W(y) &= \prod_{d \in D} w_{y(d)} \\ &= \prod_{d \in D} w_{g*x(d)} \\ &= \prod_{d \in D} w_{x(g^{-1}(d))} \qquad (2) \\ &= \prod_{d \in D} w_{x(d))} \qquad (3) \\ &= W(x) \end{aligned}$$

Note, that we can go from (2) to (3) because as d runs over D, $g^{-1}(d)$ also runs over d.

Let $\Delta = |D|$. If $g \in G$ has k_i cycles of length i then we define

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$

The **Cycle Index Polynomial** of G, C_G is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2 x_2$	$x_1^2 x_2$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$

In Example 2 with $n = 3$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^9	$x_1 x_4^2$	$x_1 x_2^4$	$x_1 x_4^2$	$x_1^3 x_2^3$	$x_1^3 x_2^3$	$x_1^3 x_2^3$	$x_1^3 x_2^3$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1 x_2^4 + 4x_1^3 x_2^3 + 2x_1 x_4^2).$$

Theorem (Polya)

$$PI = C_G\left(\sum_{c\in C} w_c, \sum_{c\in C} w_c^2, \ldots, \sum_{c\in C} w_c^{\Delta}\right).$$

Proof In Example 2, we replace x_1 by $R+B$, x_2 by R^2+B^2 and so on. When $n=2$ this gives

$$\begin{aligned} PI &= \frac{1}{8}((R+B)^4 + 3(R^2+B^2)^2 + \\ &\qquad 2(R+B)^2(R^2+B^2) + 2(R^4+B^4)) \\ &= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \end{aligned}$$

Putting $R=B=1$ gives the number of distinct colorings. Note also the formula for PI tells us that there are 2 distinct colorings using 2 reds and 2 Blues.

Proof of Polya's Theorem

Let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ be the equivalence clases of X under the relation

$$x \sim y \text{ iff } W(x) = W(y).$$

By Lemma 2, $g * x \sim x$ for all $x \in X, g \in G$ and so we can think of G acting on each X_i individually i.e. we use the fact that $x \in X_i$ implies $g * x \in X_i$ for all $i \in [m], g \in G$. We use the notation $g^{(i)} \in G^{(i)}$ when we restrict attention to X_i.

Let m_i denote the number of orbits $\nu_{X_i, G^{(i)}}$ and W_i denote the common PI of $G^{(i)}$ acting on X_i. Then

$$
\begin{aligned}
PI &= \sum_{i=1}^{m} m_i W_i \\
&= \sum_{i=1}^{m} W_i \left(\frac{1}{|G|} \sum_{g \in G} |Fix(g^{(i)})| \right) \qquad \text{by Theorem 2} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i \\
&= \frac{1}{|G|} \sum_{g \in G} W(Fix(g)) \qquad (4)
\end{aligned}
$$

Note that (4) follows from $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ iff $x \in X_i$ and $g * x = x$.

Suppose now that $ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left(\sum_{c \in C} w_c\right)^{k_1} \left(\sum_{c \in C} w_c^2\right)^{k_2} \cdots \left(\sum_{c \in C} w_c^\Delta\right)^{k_\Delta}. \qquad (5)$$

Substituting (5) into (4) yields the theorem.

To verify (5) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of g must be given the same color. A cycle of length i will then contribute a factor $\sum_{c \in C} w_c^i$ where the term w_c^i comes from the choice of color c for every element of the cycle. □