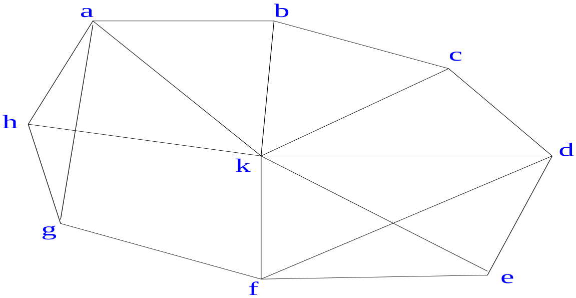
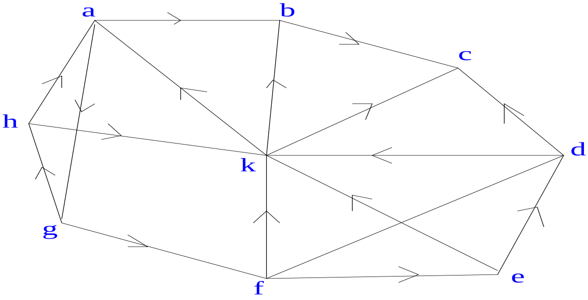
Graph Theory

Graph $G = (V, E)$.
V={vertices}, E={edges}.



V={a,b,c,d,e,f,g,h,k}
E={(a,b),(a,g),(a,h),(a,k),(b,c),(b,k),...,(h,k)} |E|=16.

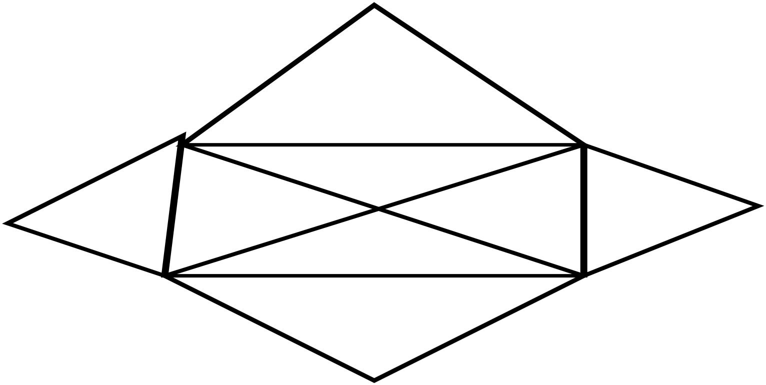
Digraph $D = (V, A)$.
V={vertices}, E={edges}.



V={a,b,c,d,e,f,g,h,k}
E={(a,b),(a,g),(h,a),(k,a),(b,c),(k,b),...,(h,k)} |E|=16.

Eulerian Graphs

Can you draw the diagram below without taking your pen off the paper or going over the same line twice?

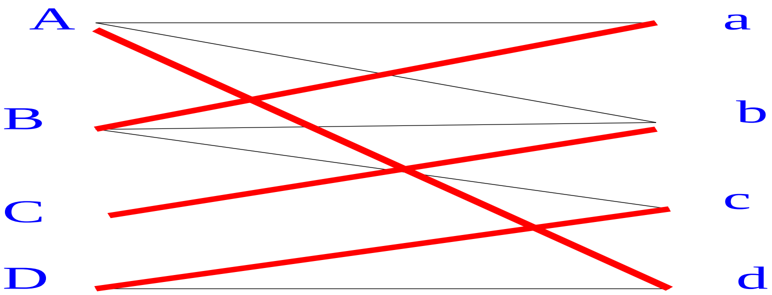


Bipartite Graphs

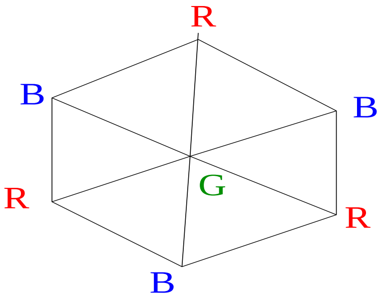
G is bipartite if $V = X \cup Y$ where X and Y are disjoint and every edge is of the form (x, y) where $x \in X$ and $y \in Y$.

In the diagram below, A,B,C,D are women and a,b,c,d are men.

There is an edge joining x and y iff x and y like each other. The thick edges form a "perfect matching" enabling everybody to be paired with someone they like. Not all graphs will have perfect matching!



Vertex Colouring

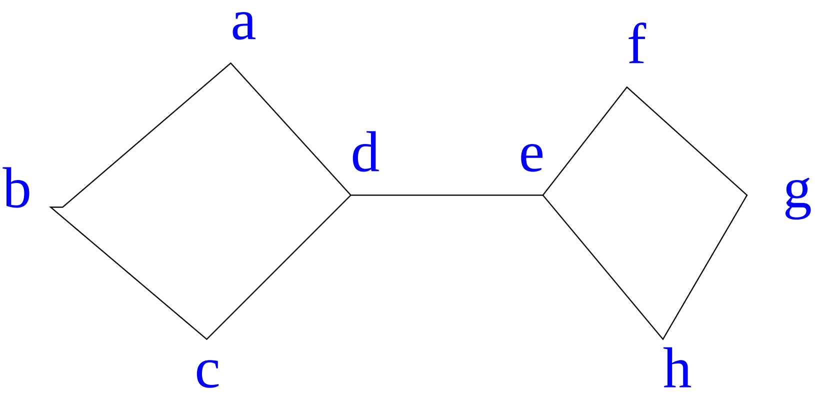


Colours {R,B,G}

Let $C = \{colours\}$. A vertex colouring of G is a map $f : V \to C$.
We say that $v \in V$ gets coloured with $f(v)$.
The colouring is *proper* iff $(a, b) \in E \Rightarrow f(a) \neq f(b)$.
The *Chromatic Number* $\chi(G)$ is the minimum number of colours in a proper colouring.

Subgraphs

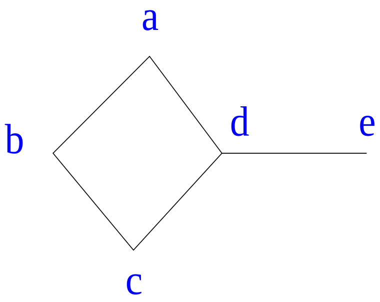
$G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
G' is a *spanning* subgraph if $V' = V$.



If $V' \subseteq V$ then

$$G[V'] = (V', \{(u,v) \in E : u,v \in V'\})$$

is the subgraph of G *induced* by V'.



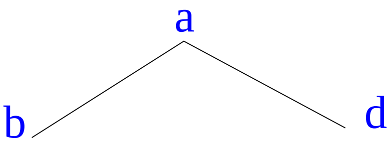
$G[\{a,b,c,d,e\}]$

Similarly, if $E_1 \subseteq E$ then $G[E_1] = (V_1, E_1)$ where

$$V_1 = \{v \in V_1 : \exists e \in E_1 \text{ such that } v \in e\}$$

is also *induced* (by E_1).

$$E_1 = \{(a,b), (a,d)\}$$

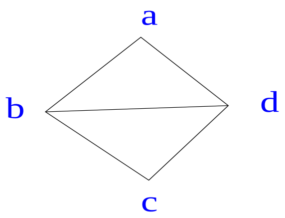


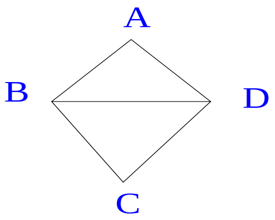
$G[E_1]$

Isomorphism

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $f : V_1 \to V_2$ such that

$$(v, w) \in E_1 \leftrightarrow (f(v), f(w)) \in E_2.$$





f(a)=A etc.

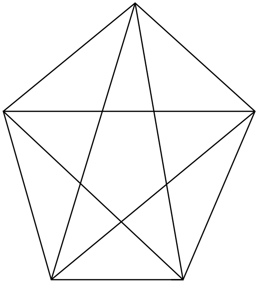
Complete Graphs

$$K_n = ([n], \{(i,j) : 1 \le i < j \le n\})$$

is the complete graph on n vertices.

$$K_{m,n} = ([m] \cup [n], \{(i,j) : i \in [m], j \in [n]\})$$

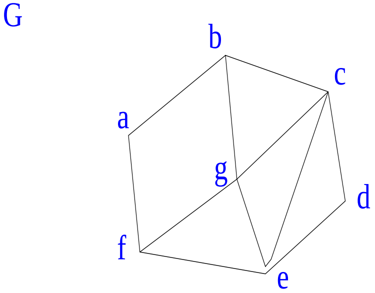
is the complete bipartite graph on $m + n$ vertices.
(The notation is a little imprecise but hopefully clear.)



K_5

Vertex Degrees

$$
\begin{aligned}
d_G(v) &= \text{degree of vertex } v \text{ in } G \\
&= \text{number of edges incident with } v \\
\delta(G) &= \min_v d_G(v) \\
\Delta(G) &= \max_v d_G(v)
\end{aligned}
$$



$d_G(a)=2$, $d_G(g)=4$ etc.

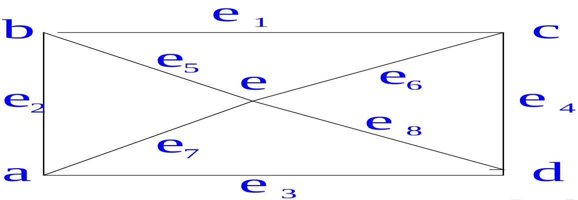
$\delta(G)=2$, $\Delta(G)=4$.

Matrices and Graphs

Incidence matrix M: $V \times E$ matrix.

$$M(v,e) = \begin{cases} 1 & v \in e \\ 0 & v \notin e \end{cases}$$

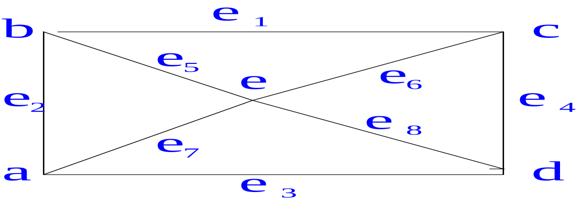
	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a		1	1				1	
b	1	1			1			
c	1			1		1		
d			1	1				1
e					1	1	1	1



Adjacency matrix A: $V \times V$ matrix.

$$A(v,w) = \begin{cases} 1 & v,w \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

	a	*b*	*c*	*d*	*e*
a		1		1	1
b	1		1		1
c		1		1	1
d	1		1		1
e	1	1	1	1	



Theorem

$$\sum_{v \in V} d_G(v) = 2|E|$$

Proof Consider the incidence matrix M. Row v has $d_G(v)$ 1's. So

$$\# \text{ 1's in matrix } M \text{ is } \sum_{v \in V} d_G(v).$$

Column e has 2 1's. So

$$\# \text{ 1's in matrix } M \text{ is } 2|E|.$$

□

Corollary

In any graph, the number of vertices of odd degree, is even.

Proof Let ODD = {odd degree vertices} and $EVEN = V \setminus ODD$.

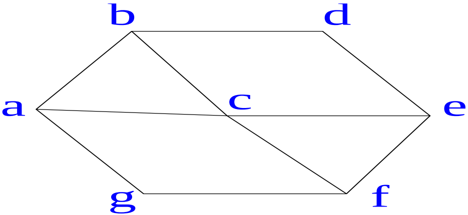
$$\sum_{v \in ODD} d(v) = 2|E| - \sum_{v \in EVEN} d(v)$$

is even.

So $|ODD|$ is even. □

Paths and Walks

$W = (v_1, v_2, \ldots, v_k)$ is a walk in G if $(v_i, v_{i+1}) \in E$ for $1 \le i < k$.
A path is a walk in which the vertices are distinct.
W_1 is a path, but W_2, W_3 are not.



W_1 = a,b,c,e,d

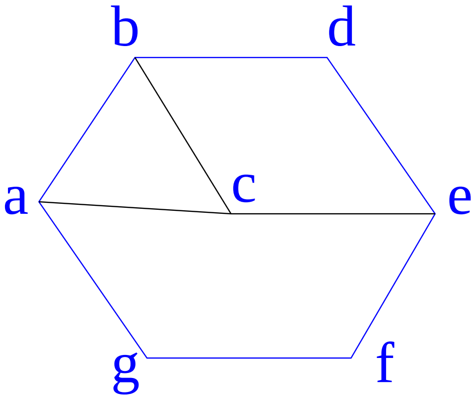
W_2 =a,b,a,c,e

W_3 =g,f,c,e,f

A walk is *closed* if $v_1 = v_k$. A *cycle* is a closed walk in which the vertices are distinct except for v_1, v_k.

b, c, e, d, b is a cycle.

b, c, a, b, d, e, c, b is not a cycle.



Theorem

Let A be the adjacency matrix of the graph $G = (V, E)$ and let $M_k = A^k$ for $k \geq 1$. Then for $v, w \in V$, $M_k(v, w)$ is the number of distinct walks of length k from v to w.

Proof We prove this by induction on k. The base case $k = 1$ is immediate.

Assume the truth of the theorem for some $k \geq 1$. For $\ell \geq 0$, let $\mathcal{P}_\ell(x, y)$ denote the set of walks of length ℓ from x to y. Let $\mathcal{P}_{k+1}(v, w; x)$ be the set of walks from v to w whose *penultimate* vertex is x. Note that

$$\mathcal{P}_{k+1}(v, w; x) \cap \mathcal{P}_{k+1}(v, w; x') = \emptyset \text{ for } x \neq x'$$

and

$$\mathcal{P}_{k+1}(v, w) = \bigcup_{x \in V} \mathcal{P}_{k+1}(v, w; x)$$

So,

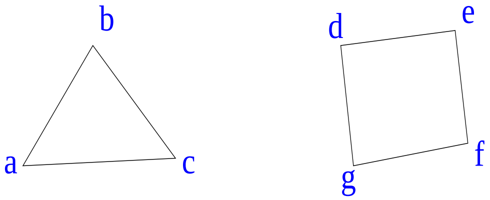
$$
\begin{aligned}
|\mathcal{P}_{k+1}(v, w)| &= \sum_{x \in V} |\mathcal{P}_{k+1}(v, w; x)| \\
&= \sum_{x \in V} |\mathcal{P}_k(v, x)| A(x, w) \\
&= \sum_{x \in V} M_k(v, x) A(x, w) \qquad \text{induction} \\
&= M_{k+1}(v, w) \qquad \text{matrix multiplication}
\end{aligned}
$$

□

Connected components

We define a relation $\sim$ on V.

$a \sim b$ iff there is a walk from a to b.



$a \sim b$ but $a \not\sim d$.

Claim: $\sim$ is an equivalence relation.

reflexivity $v \sim v$ as v is a (trivial) walk from v to v.

Symmetry $u \sim v$ implies $v \sim u$.

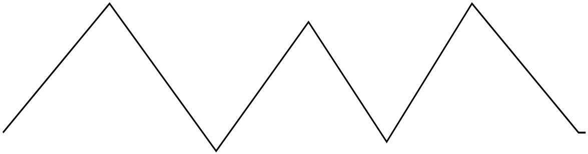
$(u = u_1, u_2 \ldots, u_k = v)$ is a walk from u to v

implies $(u_k, u_{k-1}, \ldots, u_1)$ is a walk from v to u.

Transitivity $u \sim v$ and $v \sim w$ implies $u \sim w$.
$W_1 = (u = u_1, u_2 \dots, u_k = v)$ is a walk from u to v and $W_2 = (v_1 = v, v_2, v_3, \dots, v_\ell = w)$ is a walk from v to w imples that $(W_1, W_2) = (u_1, u_2 \dots, u_k, v_2, v_3, \dots, v_\ell)$ is a walk from u to w.

The equivalence classes of $\sim$ are called *connected components*.
In general $V = C_1 \cup V_2 \cup \cdots \cup C_r$ where $C_1, C_2, \dots,$ C_r are the connected comonents.
We let $comp(G)(= r)$ be the number of components of G.
G is *connected* iff $comp(G) = 1$ i.e. there is a walk between every pair of vertices.
Thus $C_1, C_2, \dots, C_r$ induce connected subgraphs $G[C_1], \dots, G[C_r]$ of G

For a walk W we let $\ell(W)$ = no. of edges in W.



Lemma

Suppose W is a walk from vertex a to vertex b and that W minimises ℓ over all walks from a to b. Then W is a path.

Proof Suppose $W = (a = a_0, a_1, \ldots, a_k = b)$ and $a_i = a_j$ where $0 \leq i < j \leq k$. Then $W' = (a_0, a_1, \ldots, a_i, a_{j+1}, \ldots, a_k)$ is also a walk from a to b and $\ell(W') = \ell(W) - (j - i) < \ell(W)$ – contradiction. □

Corollary

If $a \sim b$ then there is a path from a to b.

So G is connected $\leftrightarrow \forall a, b \in V$ there is a path from a to b.

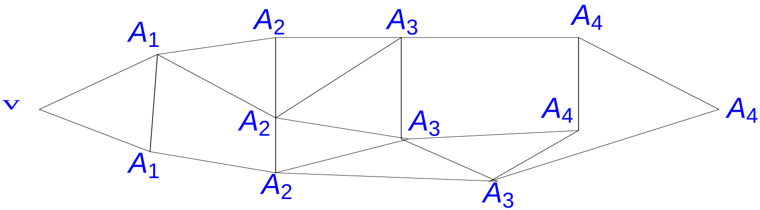
Breadth First Search – BFS

Fix $v \in V$. For $w \in V$ let

$$d(v,w) = \text{length of shortest path from } v \text{ to } w.$$

For $t = 0, 1, 2, \ldots,$ let

$$A_t = \{w \in V : \; d(v,w) = t\}.$$



$A_0 = \{v\}$ and $v \sim w \leftrightarrow d(v,w) < \infty$.

In BFS we construct $A_0, A_1, A_2, \ldots,$ by

$$A_{t+1} = \{w \notin A_0 \cup A_1 \cup \cdots \cup A_t : \exists \text{ an edge } (u, w) \text{ such that } u \in A_t\}.$$

$$\text{Note :} \quad \text{no edges } (a, b) \text{ between } A_k \text{ and } A_\ell \text{ for } \ell - k \geq 2, \text{ else } w \in A_{k+1} \neq A_\ell. \tag{1}$$

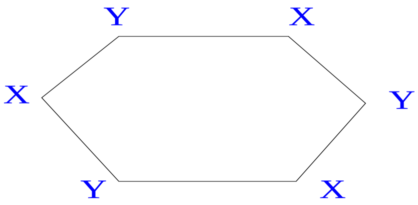
In this way we can find all vertices in the same component C as v.
By repeating for $v' \notin C$ we find another component etc.

Characterisation of bipartite graphs

Theorem

G is bipartite $\leftrightarrow$ G has no cycles of odd length.

Proof $\rightarrow$: $G = (X \cup Y, E)$.



Typical Cycle

Suppose $C = (u_1, u_2, \ldots, u_k, u_1)$ is a cycle. Suppose $u_1 \in X$. Then $u_2 \in Y, u_3 \in X, \ldots, u_k \in Y$ implies k is even.

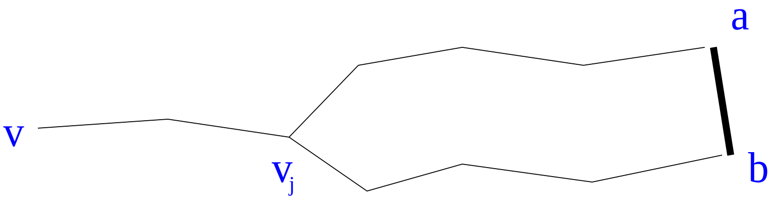
$\leftarrow$ Assume G is connected, else apply following argument to each component.
Choose $v \in V$ and construct $A_0, A_1, A_2, \ldots,$ by BFS.

$$X = A_0 \cup A_2 \cup A_4 \cup \cdots \text{ and } Y = A_1 \cup A_3 \cup A_5 \cup \cdots$$

We need only show that X and Y contain no edges and then all edges must join X and Y. Suppose X contains edge (a, b) where $a \in A_k$ and $b \in A_\ell$.
(i) If $k \neq \ell$ then $|k - \ell| \geq 2$ which contradicts (1)

(ii)

$k = \ell$:

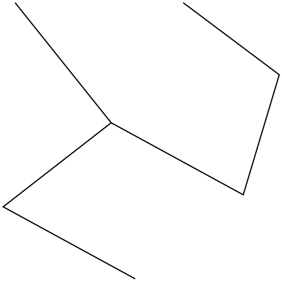
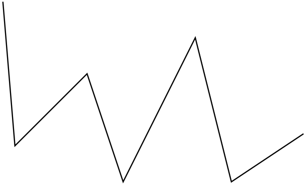


There exist paths $(v = v_0, v_1, v_2, \ldots, v_k = a)$ and $(v = w_0, w_1, w_2, \ldots, w_k = b)$.
Let $j = \max\{t : v_t = w_t\}$.

$$(v_j, v_{j+1}, \ldots, v_k, w_k, w_{k-1}, \ldots, w_j)$$

is an odd cycle – length $2(k-j)+1$ – contradiction. □

Trees

A *tree* is a graph which is

(a) Connected and
(b) has no cycles (*acyclic*).

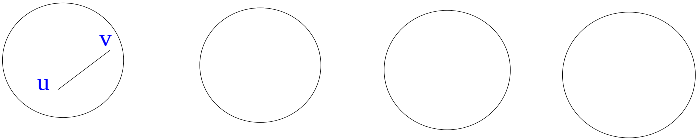
Lemma

Let the components of G *be* $C_1, C_2, \ldots, C_r$, *Suppose* $e = (u, v) \notin E$, $u \in C_i$, $v \in C_j$.

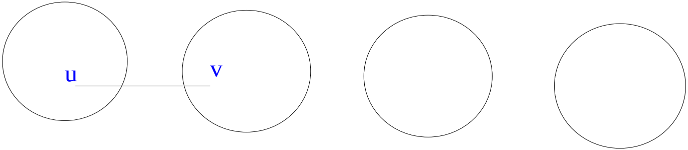
(a) $i = j \Rightarrow comp(G + e) = comp(G)$.

(b) $i \neq j \Rightarrow comp(G + e) = comp(G) - 1$.

(a)



(b)



Proof Every path P in $G + e$ which is not in G must contain e. Also,

$$comp(G + e) \leq comp(G).$$

Suppose

$$(x = u_0, u_1, \ldots, u_k = u, u_{k+1} = v, \ldots, u_\ell = y)$$

is a path in $G + e$ that uses e. Then clearly $x \in C_i$ and $y \in C_j$.
(a) follows as now no new relations $x \sim y$ are added.
(b) Only possible new relations $x \sim y$ are for $x \in C_i$ and $y \in C_j$.
But $u \sim v$ in $G + e$ and so $C_i \cup C_j$ becomes (only) new component. □

Lemma

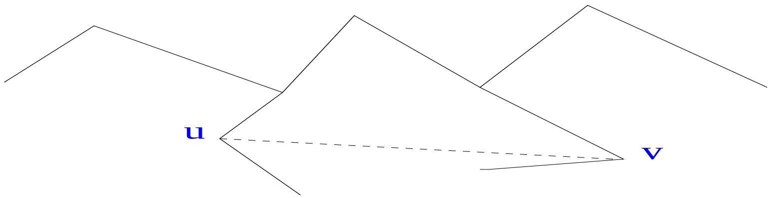
$G = (V, E)$ *is acyclic* (forest) *with (tree) components* $C_1, C_2, \ldots, C_k$. $|V| = n$. $e = (u, v) \notin E$, $u \in C_i$, $v \in C_j$.

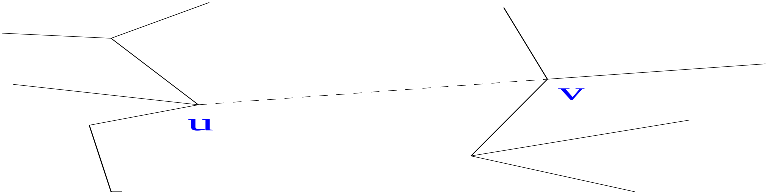
(a) $i = j \Rightarrow G + e$ *contains a cycle.*

(b) $i \neq j \Rightarrow G + e$ *is acyclic and has one less component.*

(c) G *has* $n - k$ *edges.*

(a) $u, v \in C_i$ implies there exists a path
$(u = u_0, u_1, \ldots, u_\ell = v)$ in G.
So $G + e$ contains the cycle $u_0, u_1, \ldots, u_\ell, u_0$.

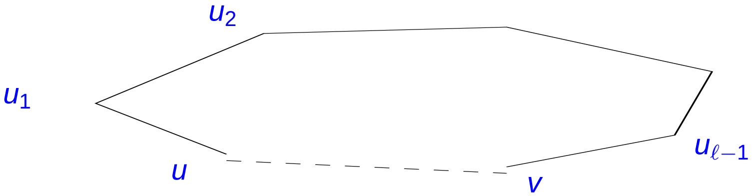




(b) Suppose $G + e$ contains the cycle C. $e \in C$ else C is a cycle of G.

$$C = (u = u_0, u_1, \ldots, u_\ell = v, u_0).$$

But then G contains the path $(u_0, u_1, \ldots, u_\ell)$ from u to v – contradiction.



Drop in number of components follows from previous Lemma.

The rest follows from

(c) Suppose $E = \{e_1, e_2, \ldots, e_r\}$ and
$G_i = (V, \{e_1, e_2, \ldots, e_i\})$ for $0 \leq i \leq r$.

Claim: G_i has $n - i$ components.

Induction on i.

$i = 0$: G_0 has no edges.

$i > 0$: G_{i-1} is acyclic and so is G_i. It follows from part (a) that e_i joins vertices in distinct components of G_{i-1}. It follows from (b) that G_i has one less component than G_{i-1}.

End of proof of claim

Thus $r = n - k$ (we assumed G had k components). □

Corollary

If a tree T has n vertices then

(a) *It has $n-1$ edges.*

(b) *It has at least 2 vertices of degree 1, ($n \geq 2$).*

Proof (a) is part (c) of previous lemma. $k=1$ since T is connnected.

(b) Let s be the number of vertices of degree 1 in T. There are no vertices of degree 0 – these would form separate components. Thus

$$2n-2=\sum_{v\in V} d_T(v) \geq 2(n-s)+s.$$

So $s \geq 2$. □

Theorem

Suppose $|V| = n$ and $|E| = n - 1$. The following three statements become equivalent.

(a) G *is connected.*

(b) G *is acyclic.*

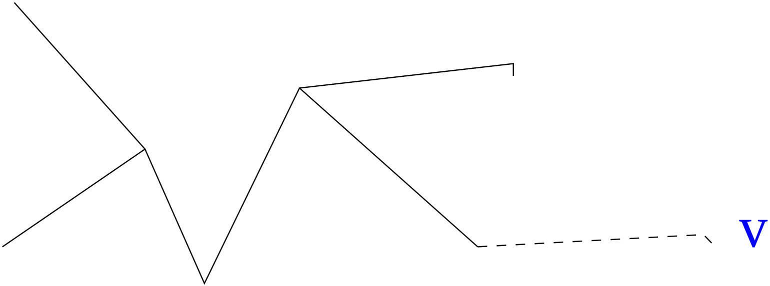
(c) G *is a tree.*

Let $E = \{e_1, e_2, \ldots, e_{n-1}\}$ and $G_i = (V, \{e_1, e_2, \ldots, e_i\})$ for $0 \leq i \leq n - 1$.

$(a) \Rightarrow (b)$: G_0 has n components and G_{n-1} has 1 component. Addition of each edge e_i must reduce the number of components by 1. Thus G_{i-1} acyclic implies G_i is acyclic. (b) follows as G_0 is acyclic.
$(b) \Rightarrow (c)$: We need to show that G is connected. Since G_{n-1} is acyclic, $comp(G_i) = comp(G_{i-1}) - 1$ for each i. Thus $comp(G_{n-1}) = 1$.
$(c) \Rightarrow (a)$: trivial.

Corollary

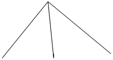
If v is a vertex of degree 1 in a tree T then $T - v$ is also a tree.



Proof Suppose T has n vertices and $n - 1$ edges. Then $T - v$ has $n - 1$ vertices and $n - 2$ edges. It acyclic and so must be a tree. □

How many trees? – Cayley’s Formula

n=4



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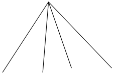
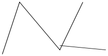
n=5



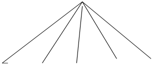
5



60

60

n=6



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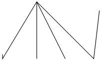
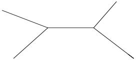


120

360



90

360

360

Prüfer's Correspondence

There is a 1-1 correspondence ϕ_V between spanning trees of K_V (the complete graph with vertex set V) and sequences V^{n-2}. Thus for $n \geq 2$

$$\tau(K_n) = n^{n-2} \qquad \text{Cayley's Formula.}$$

Assume some arbitrary ordering $V = \{v_1 < v_2 < \cdots < v_n\}$.

$\phi_V(T)$:

begin

$T_1 := T;$

for $i = 1$ **to** $n-2$ **do**

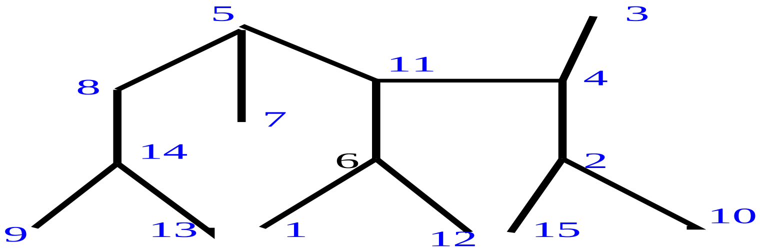
begin

$s_i :=$ neighbour of least leaf ℓ_i of T_i.

$T_{i+1} = T_i - \ell_i$.

end $\qquad \phi_V(T) = s_1 s_2 \ldots s_{n-2}$

end

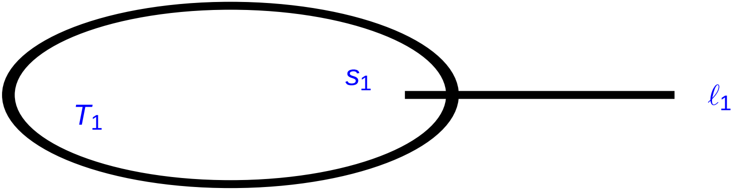


6,4,5,14,2,6,11,14,8,5,11,4,2

Lemma

$v \in V(T)$ appears exactly $d_T(v) - 1$ times in $\phi_V(T)$.

Proof Assume $n = |V(T)| \geq 2$. By induction on n.
$n = 2$: $\phi_V(T) = \Lambda$ = empty string.
Assume $n \geq 3$:



$\phi_V(T) = s_1\phi_{V_1}(T_1)$ where $V_1 = V - \{s_1\}$.
s_1 appears $d_{T_1}(s_1) - 1 + 1 = d_T(s_1) - 1$ times – induction.
$v \neq s_1$ appears $d_{T_1}(v) - 1 = d_T(v) - 1$ times – induction. □

Construction of ϕ_V^{-1}

Inductively assume that for all $|X| < n$ there is an inverse function ϕ_X^{-1}. (True for $n = 2$).
Now define ϕ_V^{-1} by

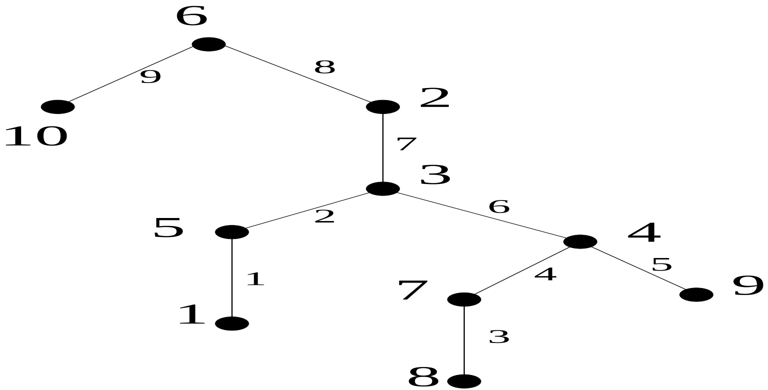
$$\phi_V^{-1}(s_1 s_2 \ldots s_{n-2}) = \phi_{V_1}^{-1}(s_2 \ldots s_{n-2}) \text{ plus edge } s_1\ell_1,$$

where $\ell_1 = \min\{s \in V : s \notin \{s_1, s_2, \ldots s_{n-2}\}\}$ and $V_1 = V - \{\ell_1\}$. Then

$$\begin{aligned} \phi_V(\phi_V^{-1}(s_1 s_2 \ldots s_{n-2})) &= s_1 \phi_{V_1}(\phi_{V_1}^{-1}(s_2 \ldots s_{n-2})) \\ &= s_1 s_2 \ldots s_{n-2}. \end{aligned}$$

Thus ϕ_V has an inverse and the correspondence is established.

$n = 10$
$s = 5, 3, 7, 4, 4, 3, 2, 6$.



Number of trees with a given degree sequence

Corollary

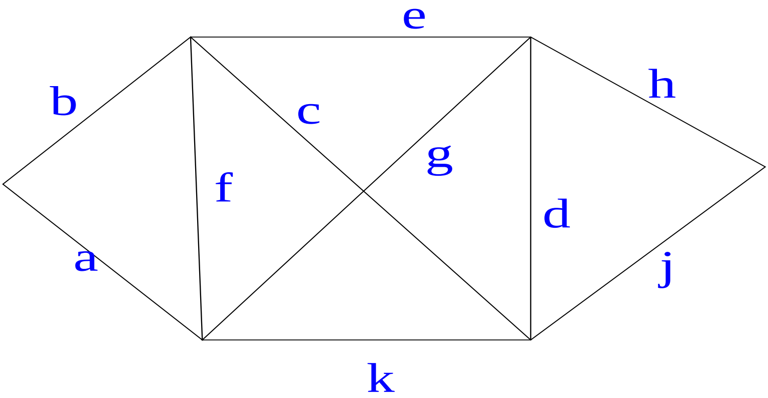
If $d_1 + d_2 + \cdots + d_n = 2n - 2$ *then the number of spanning trees of* K_n *with degree sequence* $d_1, d_2, \ldots, d_n$ *is*

$$\binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}.$$

Proof From Prüfer's correspondence this is the number of sequences of length $n - 2$ in which 1 appears $d_1 - 1$ times, 2 appears $d_2 - 1$ times and so on. □

Eulerian Graphs

An *Eulerian cycle* of a graph $G = (V, E)$ ia closed walk which uses each edge $e \in E$ exactly once.

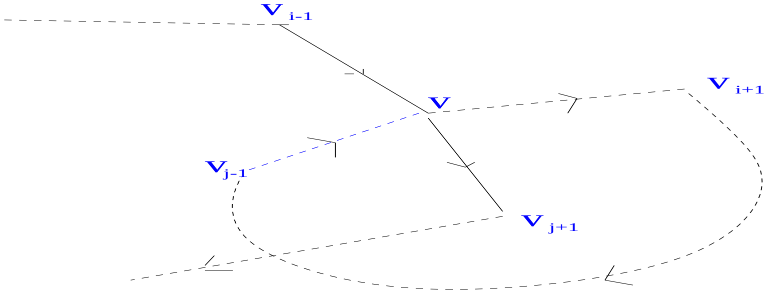


The walk using edges $a, b, c, d, e, f, g, h, j, k$ in this order is an Eulerian cycle.

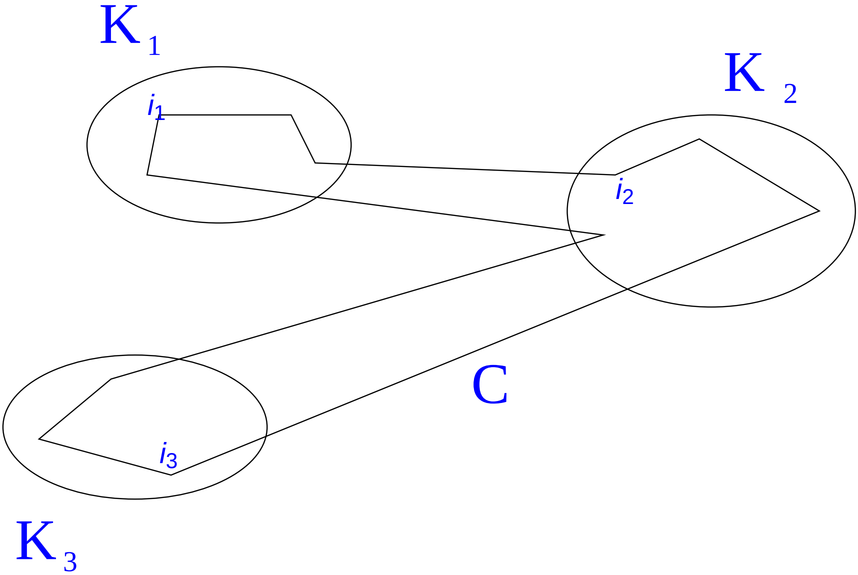
Theorem

A connected graph is Eulerian i.e. has an Eulerian cycle, iff it has no vertex of odd degree.

Proof Suppose $W = (v_1, v_2, \ldots, v_m, v_1)$ ($m = |E|$) is an Eulerian cycle. Fix $v \in V$. Whenever W visits v it enters through a new edge and leaves through a new edge. Thus each visit requires 2 new edges. Thus the degree of v is even.



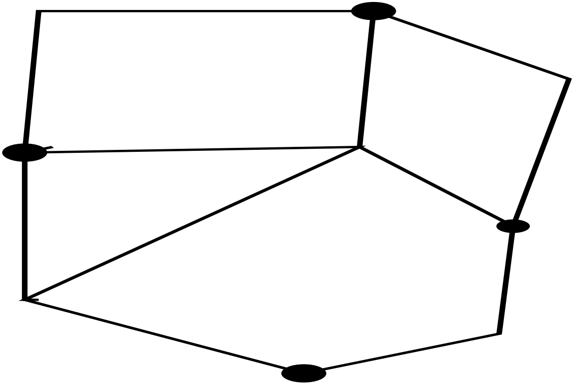
The converse is proved by induction on $|E|$. The result is true for $|E| = 3$. The only possible graph is a triangle.
Assume $|E| \geq 4$. G is not a tree, since it has no vertex of degree 1. Therefore it contains a cycle C. Delete the edges of C. The remaining graph has components $K_1, K_2, \ldots, K_r$.
Each K_i is connected and is of even degree – deleting C removes 0 or 2 edges incident with a given $v \in V$. Also, each K_i has strictly less than $|E|$ edges. So, by induction, each K_i has an Eulerian cycle, C_i say.
We create an Eulerian cycle of G as follows: let $C = (v_1, v_2, \ldots, v_s, v_1)$. Let v_{i_t} be the first vertex of C which is in K_t. Assume w.l.o.g. that $i_1 < i_2 < \cdots < i_r$.
$W = (v_1, v_2, \ldots, v_{i_1}, C_1, , v_{i_1}, \ldots, v_{i_2}, C_2, v_{i_2}, \ldots, v_{i_r}, C_r, v_{i_r}, \ldots, v_1)$
is an Eulerian cycle of G. □



K_1
i_1
K_2
i_2
C
i_3
K_3

Independent sets and cliques

$S \subseteq V$ is *independent* if no edge of G has both of its endpoints in S.



$\alpha(G)$=maximum size of an independent set of G.

Theorem

If graph G has n vertices and m edges then

$$\alpha(G) \geq \frac{n^2}{2m+n}.$$

Note that this says that $\alpha(G)$ is at least $\frac{n}{d+1}$ where d is the average degree of G.

Proof Let $\pi(1), \pi(2), \ldots, \pi(\nu)$ be an arbitrary permutation of V. Let $N(v)$ denote the set of neighbours of vertex v and let

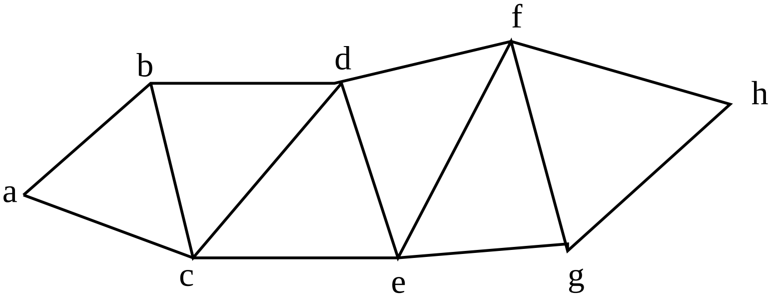
$$I(\pi) = \{v : \pi(w) > \pi(v) \text{ for all } w \in N(v)\}.$$

Claim

I is an independent set.

Proof of Claim 1

Suppose $w_1, w_2 \in I(\pi)$ and $w_1 w_2 \in E$. Suppose $\pi(w_1) < \pi(w_2)$. Then $w_2 \notin I(\pi)$ — contradiction. □



	a	b	c	d	e	f	g	h	I
π_1	c	b	f	h	a	g	e	d	$\{c, f\}$
π_2	g	f	h	d	e	a	b	c	$\{g, a\}$

Claim

If π is a random permutation then

$$\mathbf{E}(|I|) = \sum_{v \in V} \frac{1}{d(v)+1}.$$

Proof: Let $\delta(v) = \begin{cases} 1 & v \in I \\ 0 & v \notin I \end{cases}$

Thus

$$\begin{aligned} |I| &= \sum_{v \in V} \delta(v) \\ \mathbf{E}(|I|) &= \sum_{v \in V} \mathbf{E}(\delta(v)) \\ &= \sum_{v \in V} \mathbf{Pr}(\delta(v) = 1). \end{aligned}$$

Now $\delta(v) = 1$ iff v comes before all of its neighbours in the order π. Thus

$$\mathbf{Pr}(\delta(v) = 1) = \frac{1}{d(v) + 1}$$

and the claim follows. □

Thus there exists a π such that

$$|I(\pi)| \geq \sum_{v \in V} \frac{1}{d(v) + 1}$$

and so

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1}.$$

We finish the proof of the theorem by showing that

$$\sum_{v \in V} \frac{1}{d(v)+1} \geq \frac{n^2}{2m+n}.$$

This follows from the following claim by putting $x_v = d(v) + 1$ for $v \in V$.

Claim

If $x_1, x_2, \ldots x_k > 0$ *then*

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \geq \frac{k^2}{x_1 + x_2 + \cdots + x_k}. \tag{2}$$

Proof

Multiplying (2) by $x_1 + x_2 + \cdots + x_k$ and subtracting k from both sides we see that (2) is equivalent to

$$\sum_{1 \le i < j \le k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right) \ge k(k-1). \tag{3}$$

But for all $x, y > 0$

$$\frac{x}{y} + \frac{y}{x} \ge 2$$

and (3) follows. □

Corollary

If G contains no clique of size k then

$$m \leq \frac{(k-2)n^2}{2(k-1)}$$

For example, if G contains no triangle then $m \leq n^2/4$.

Proof Let $\bar{G}$ be the *complement* of G i.e. $G + \bar{G} = K_n$.

By assumption

$$k - 1 \geq \alpha(\bar{G}) \geq \frac{n^2}{n(n-1) - 2m + n}.$$

□

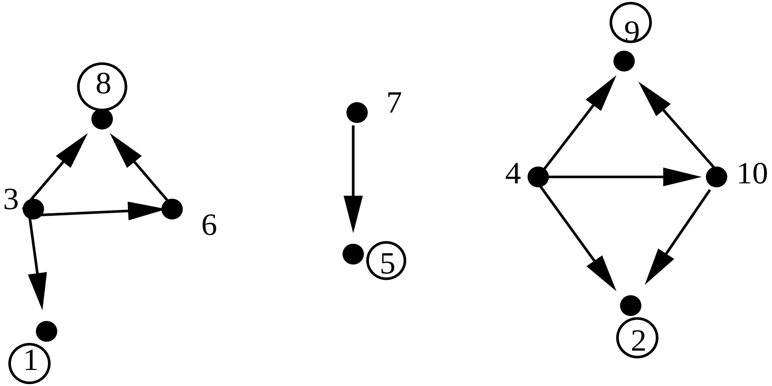
Parallel searching for the maximum – Valiant

We have n processors and n numbers $x_1, x_2, \ldots, x_n$. In each round we choose n pairs i, j and compare the values of x_i, x_j. The set of pairs chosen in a round can depend on the results of previous comparisons.

Aim: find i^* such that $x_{i^*} = \max_i x_i$.

Claim

For any algorithm there exists an input which requires at least $\frac{1}{2}\log_2\log_2 n$ *rounds.*



Suppose that the first round of comparisons involves comparing x_i, x_j for edge ij of the above graph and that the arrows point to the larger of the two values. Consider the independent set $\{1, 2, 5, 8, 9\}$. These are the indices of the 5 largest elements, but their relative order can be arbitrary since there is no implied relation between their values.

Let $C(a,b)$ be the maximum number of rounds needed for a processors to compute the maximum of b values in this way.

Lemma

$$C(a,b) \geq 1 + C\left(a, \left\lceil \frac{b^2}{2a+b} \right\rceil\right).$$

Proof The set of b comparisons defines a b-edge graph G on a vertices where comparison of x_i, x_j produces an edge ij of G. Now,

$$\alpha(G) \geq \left\lceil \frac{b}{\frac{2a}{b}+1} \right\rceil = \left\lceil \frac{b^2}{2a+b} \right\rceil.$$

For any independent set I it is always possible to define values for $x_1, x_2, \ldots, x_a$ such I is the index set of the $|I|$ largest values and so that the comparisons do not yield any information about the ordering of the elements $x_i, i \in I$.
Thus after one round one has the problem of finding the maximum among $\alpha(G)$ elements. □
Now define the sequence $c_0, c_1, \ldots$ by $c_0 = n$ and

$$c_{i+1} = \left\lceil \frac{c_i^2}{2n + c_i} \right\rceil.$$

It follows from the previous lemma that

$$c_k \geq 2 \text{ implies } C(n,n) \geq k+1.$$

Claim 4 now follows from

Claim

$$c_i \geq \frac{n}{3^{2^i - 1}}.$$

By induction on i. Trivial for $i = 0$. Then

$$\begin{aligned} c_{i+1} &\geq \frac{n^2}{3^{2^{i+1}-2}} \times \frac{1}{2n + \frac{n}{3^{2^i-1}}} \\ &= \frac{n}{3^{2^{i+1}-1}} \times \frac{3}{2 + \frac{1}{3^{2^i-1}}} \\ &\geq \frac{n}{3^{2^{i+1}-1}}. \end{aligned}$$

□

We found an upper bound on the number of edges $m \leq n^2/4$ on graphs without triangles. We find a smaller bound if we exclude cycles of length four.

Theorem

If G contains no cycles of length four then $m \leq (n^{3/2} + n)/2$.

Proof We count in two ways the number N of paths x, y, z of length two.

For an unordered pair x, z there can be at most one y such that x, y, z forms a path. Otherwise G will contain a C_4. Thus

$$N \leq \binom{n}{2}.$$

Let $d(y)$ denote the degree of z for $y \in V$. In which case there are $\binom{d(y)}{2}$ choices of x, z to make a path x, y, z.

Thus

$$\begin{aligned} N &= \sum_{y \in V} \binom{d(y)}{2} \\ &= -m + \frac{1}{2} \sum_{y \in V} d(y)^2 \\ &\geq -m + \frac{n}{2} \left(\frac{2m}{n} \right)^2 \end{aligned}$$

Thus

$$\frac{n^2 - n}{2} \geq -m + \frac{2m^2}{n}.$$

We re-arrange to give

$$m^2 - \frac{n}{2}m - \frac{n^3 - n^2}{4} \leq 0$$

or

$$\left(m - \frac{n}{4}\right)^2 - \frac{n^3}{4} - \frac{n^2}{16} \leq 0$$

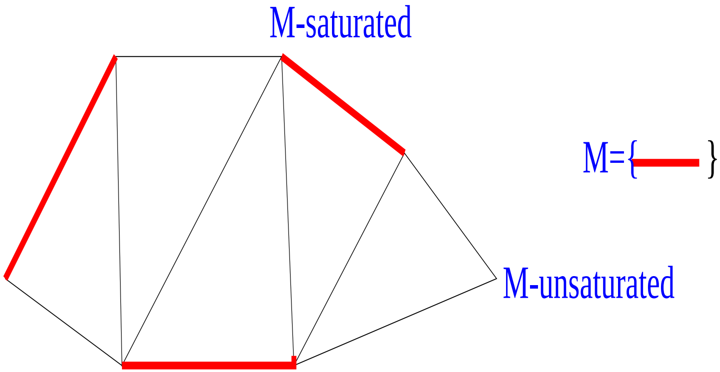
which implies that

$$m - \frac{n}{4} \leq \frac{n^{3/2}}{2} + \frac{n}{4}.$$

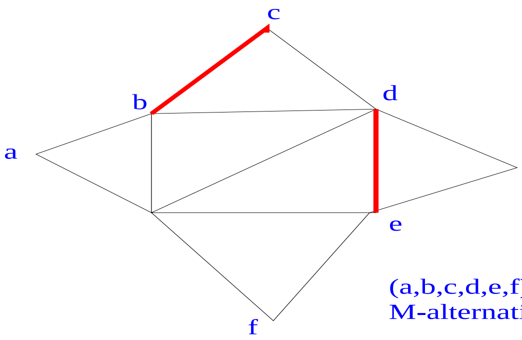
□

Matchings

A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.



M-alternating path



(a,b,c,d,e,f) is an M-alternating path

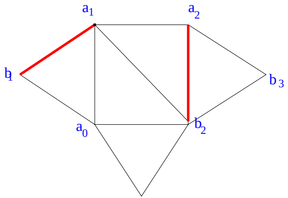
An M-alternating path joining 2 M-unsaturated vertices is called an M-augmenting path.

M is a *maximum* matching of G if no matching M' has more edges.

Theorem

M is a maximum matching iff M admits no M-augmenting paths.

Proof Suppose M has an augmenting path
$P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where
$e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and
$f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$

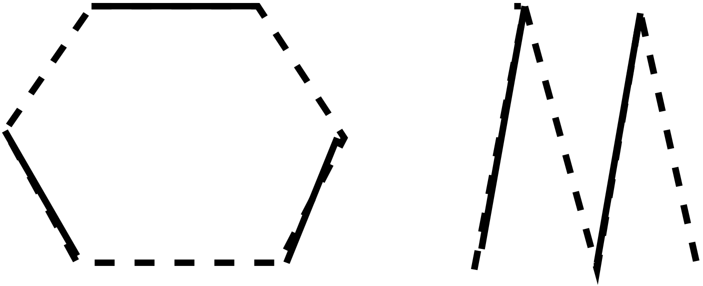
- $|M'| = |M| + 1$.
- M' is a matching

For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

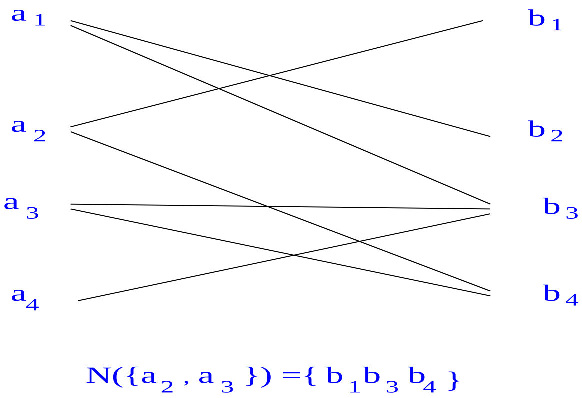
Suppose M is not a maximum matching and $|M'| > |M|$.
Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.
Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



(a) (b)

Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.
For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a,b) \in E\}$.



Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Systems of Distinct Representatives

Let $S_1, S_2, \ldots, S_m$ be arbitrary sets. A set $s_1, s_2, \ldots, s_m$ of m disitinct elements is a system of distinct representatives if $s_i \in S_i$ for $i = 1, 2, \ldots, m$.

For example $\{1, 2, 4\}$ is a system of distinct representatives for $\{1, 2, 3\}, \{2, 5, 6\}, \{2, 4, 5\}$.

Now define the bipartite graph G with vertex bipartition $[m], S$ where $S = \bigcup_{i=1}^{m} S_i$ and an edge (i, s) iff $s \in S_i$.

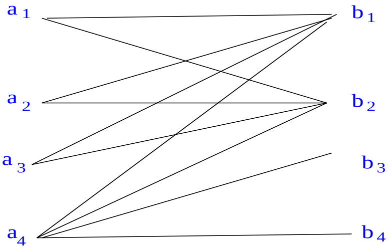
Then $S_1, S_2, \ldots, S_m$ has a system of distinct representatives iff G has a matching of size m.

Hall's Theorem

Theorem

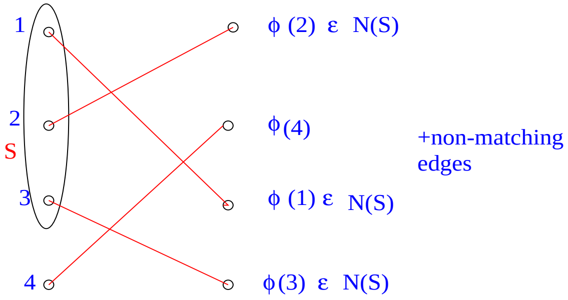
G contains a matching of size $|A|$ iff

$$|N(S)| \geq |S| \qquad \forall S \subseteq A. \tag{4}$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

: Suppose $M = \{(a, \phi(a)) : a \in A\}$ saturates A.



$$\begin{aligned} |N(S)| &\geq |\{\phi(s) : s \in S\}| \\ &= |S| \end{aligned}$$

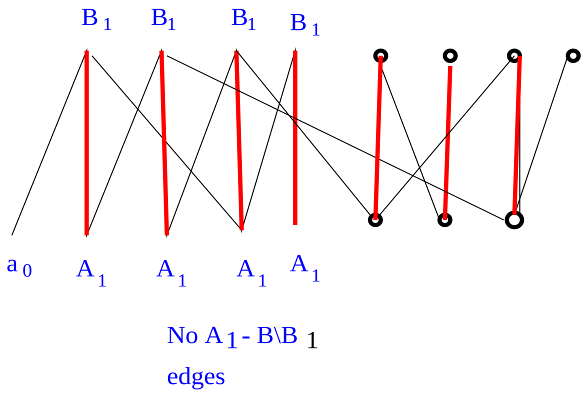
and so (4) holds.

If: Let $M = \{(a, \phi(a)) : a \in A'\}$ ($A' \subseteq A$) is a maximum matching. Suppose $a_0 \in A$ is M-unsaturated. We show that (4) fails.

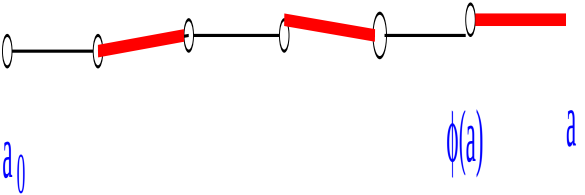
Let

$A_1 = \{a \in A :$ such that a is reachable from a_0 by an M-alternating path.$\}$

$B_1 = \{b \in B :$ such that b is reachable from a_0 by an M-alternating path.$\}$

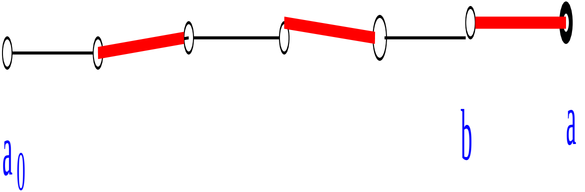


- B_1 is M-saturated else there exists an M-augmenting path.
- If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.



- If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.

So $|B_1| = |A_1| - 1$. • $N(A_1) \subseteq B_1$



So $|N(A_1)| = |A_1| - 1$ and (4) fails to hold.

Marriage Theorem

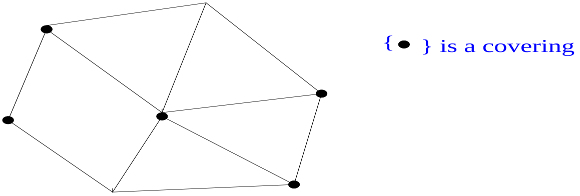
Theorem

Suppose $G = (A \cup B, E)$ is k-regular. ($k \geq 1$) i.e. $d_G(v) = k$ for all $v \in A \cup B$. Then G has a perfect matching.

Proof $k|A| = |E| = k|B|$ and so $|A| = |B|$.
Suppose $S \subseteq A$. Let m be the number of edges incident with S. Then $k|S| = m \leq k|N(S)|$. So (4) holds and there is a matching of size $|A|$ i.e. a perfect matching.

Edge Covers

A set of vertices $X \subseteq V$ is a *covering* of $G = (V, E)$ if every edge of E contains at least one endpoint in X.



Lemma

If X is a covering and M is a matching then $|X| \geq |M|$.

Proof Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. □

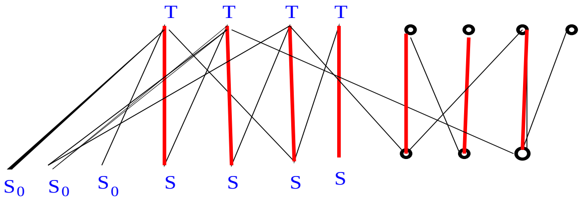
Konig's Theorem

Let $\mu(G)$ be the maximum size of a matching.
Let $\beta(G)$ be the minimum size of a covering.
Then $\mu(G) \leq \beta(G)$.

Theorem

If G *is* bipartite *then* $\mu(G) = \beta(G)$.

Proof Let M be a maximum matching.
Let S_0 be the M-unsaturated vertices of A.
Let $S \supseteq S_0$ be the A-vertices which are reachable from S by M-alternating paths.
Let T be the M-neighbours of $S \setminus S_0$.



Let $X = (A \setminus S) \cup T$.

• $|X| = |M|$.

$|T| = |S \setminus S_0|$. The remaining edges of M cover $A \setminus S$ exactly once.

• X is a cover.

There are no edges (x, y) where $x \in S$ and $y \in B \setminus T$.
Otherwise, since y is M-saturated (no M-augmenting paths) the M-neightbour of y would have to be in S, contradicting $y \notin T$. □