NETWORK FLOWS

A *network* consists of a loopless digraph $D = (V, A)$ plus a function $c : A \to \mathbf{R}_+$. Here $c(x, y)$ for $(x, y) \in A$ is the *capacity* of the edge (x, y).

We use the following notation: if $\phi : A \to \mathbf{R}$ and S, T are (not necessarily disjoint) subsets of V then

$$\phi(S, T) = \sum_{\substack{x \in S \\ y \in T}} \phi(x, y).$$

Let s, t be distinct vertices. An $s - t$ flow is a function $f : A \to \mathbf{R}$ such that

$$f(v, V \setminus \{v\}) = f(V \setminus \{v\}, v) \quad \text{for all } v \neq s, t.$$

In words: flow into v equals flow out of v.

An $s-t$ flow is *feasible* if

$$0 \leq f(x,y) \leq c(x,y) \quad \text{for all } (x,y) \in A.$$

An $s-t$ *cut* is a partition of V into two sets $S, \bar{S}$ such that $s \in S$ and $t \in \bar{S}$.

The *value* v_f of the flow f is given by

$$v_f = f(s, V \setminus \{s\}) - f(V \setminus \{s\}, s).$$

Thus v_f is the net flow leaving s.

The *capacity* of the cut $S : \bar{S}$ is equal to $c(S, \bar{S})$.

Max-Flow Min-Cut Theorem

Theorem

$$\max v_f = \min c(S, \bar{S})$$

where the maximum is over feasible $s - t$ flows and the minimum is over $s - t$ cuts.

Proof We observe first that

$$\begin{aligned} f(S, \bar{S}) - f(\bar{S}, S) &= (f(S, V) - f(S, S)) - (f(V, S) - f(S, S)) \\ &= f(S, V) - f(V, S) \\ &= v_f + \sum_{v \in S \setminus \{s\}} (f(v, V) - f(V, v)) \\ &= v_f. \end{aligned}$$

So,

$$v_f \leq f(S, \bar{S}) \leq c(S, \bar{S}).$$

This implies that

$$\max v_f \le \min c(S, \bar{S}). \tag{1}$$

Given a flow f we define a *flow augmenting path* P to be a sequence of distinct vertices $x_0 = s, x_1, x_2, \ldots, x_k = t$ such that for all i, either

- F1 $(x_i, x_{i+1}) \in A$ and $f(x_i, x_{i+1}) < c(x_i, x_{i+1})$, or
- F2 $(x_{i+1}, x_i) \in A$ and $f(x_{i+1}, x_i) > 0$.

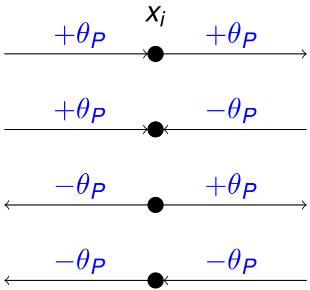
If P is such a sequence, then we define $\theta_P > 0$ to be the minimum over i of $c(x_i, x_{i+1}) - f(x_i, x_{i+1})$ (Case (F1)) and $f(x_{i+1}, x_i)$ (Case (F2)).

Claim 1: f is a maximum value flow, iff there are no flow augmenting paths.

Proof If P is flow augmenting then define a new flow f' as follows:

1. $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \theta_P$ or
2. $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \theta_P$
3. For all other edges, (x, y), we have $f'(x, y) = f(x, y)$.

We can see that the flow stays balanced at x_i.



We can see then that if there is a flow augmenting path then the new flow satisfies

$$v_{f'} = v_f + \theta_P > v_f.$$

Let S_f denote the set of vertices v for which there is a sequence $x_0 = s, x_1, x_2, \ldots, x_k = v$ which satisfies F1, F2 of the definition of flow augmenting paths.

If $t \in S_f$ then the associated sequence defines a flow augmenting path. So, assume that $t \notin S_f$. Then we have,

1. $s \in S_f$.
2. If $x \in S_f, y \in \bar{S}_f, (x, y) \in A$ then $f(x, y) = c(x, y)$, else we would have $y \in S_f$.
3. If $x \in S_f, y \in \bar{S}_f, (y, x) \in A$ then $f(y, x) = 0$, else we would have $y \in S_f$.

We therefore have

$$\begin{aligned} v_f &= f(S_f, \bar{S}_f) - f(\bar{S}_f, S) \\ &= c(S, \bar{S}_f). \end{aligned}$$

We see from this and (1) that f is a flow of maximum value and that the cut $S_f : \bar{S}_f$ is of minimum capacity.

This finishes the proof of Claim 1 and the Max-Flow Min-Cut theorem.

Note also that we can construct S_f by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds. (A simple inductive argument based on sequence length shows that all of S_f is constructed in this way.)

Note also that we can construct S_f by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds.

This defines an algorithm for finding a maximum flow. The construction either finishes with $t \in S_f$ and we can augment the flow.

Or, we find that $t \notin S_f$ and we have a maximum flow.

Note, that if all the capacities $c(x, y)$ are integers and we start with the all zero flow then we find that θ_f is always a positive integer (formally one can use induction to verify this).

It follows that in this case, there is always a maximum flow that only takes integer values on the edges.

Hall's Theorem.

Let $G = (A, B, E)$ be a bipartite graph with $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. A matching M is a set of edges that meets each vertex at most once. A matching is perfect if it meets each vertex.

Hall's theorem:

Theorem

G contains a perfect matching iff $|N(S)| \geq |S|$ for all $S \subseteq A$.

Here $N(S) = \{b \in B : \exists a \in A \text{ s.t. } \{a, b\} \in E\}$.

Define a digraph Γ by adding vertices $s, t \notin A \cup B$. Then add edges (s, a_i) and (b_i, t) of capacity 1 for $i = 1, 2, \ldots, n$. Orient the edges E for A to B and give them capacity ∞.

G has a matching of size m iff there is an $s-t$ flow of value m. An $s-t$ cut $X : \bar{X}$ has capacity

$$|A \setminus X| + |B \cap X| + |\{a \in X \cap A, b \in B \setminus X : \{a, b\} \in E\}| \times \infty.$$

It follows that to find a minimum cut, we need only consider X such that

$$\{a \in X \cap A, b \in B \setminus X : \{a, b\} \in E\} = \emptyset. \qquad (2)$$

For such a set, we let $S = A \cap X$ and $T = X \cap B$. Condition (2) means that $T \supseteq N(S)$. The capacity of $X : \bar{X}$ is now $(n - |S|) + |T|$ and for a fixed S this is minimised for $T = N(S)$.

Thus, by the Max-Flow Min-Cut theorem

$$\max\{|M|\} = \min_X\{c(X : \bar{X})\} = \min_S\{n - |S| + |N(S)\}.$$

This implies Hall's theorem.

Graph orientation problem

Let $G = (V, E)$ be a graph. When is it possible to orient the edges of G to create a digraph $\Gamma = (V, A)$ so that every vertex has out-degree at least d. We say that G is d-orientable.

Theorem

G is d-orientable iff

$$|\{e \in E : e \cap S \neq \emptyset\}| \geq d|S| \text{ for all } S \subseteq V. \tag{3}$$

Proof If G is d-orientable then

$$|\{e \in E : e \cap S \neq \emptyset\}| \geq |\{(x, y) \in A : x \in S\}| \geq d|S|.$$

Suppose now that (3) holds. Define a network D as follows; the vertices are s, t, V, E – yes, D has a vertex for each edge of G.

There is an edge of capacity d from s to each $v \in V$ and an edge of capacity one from each $e \in E$ to t. There is an edge of infinite capacity from $v \in V$ to each edge e that contains v.

Consider an integer flow f. Suppose that $e = \{v, w\} \in E$ and $f(e, t) = 1$. Then either $f(v, e) = 1$ or $f(w, e) = 1$. In the former we interpret this as orienting the edge e from v to w and in the latter from w to v.

Under this interpretation, G is d-orientable iff D has a flow of value $d|V|$.

Let $X : \bar{X}$ be an $s - t$ cut in N. Let $S = X \cap V$ and $T = X \cap E$.

To have a finite capacity, there must be no $x \in S$ and $e \in E \setminus T$ such that $x \in e$.

So, the capacity of a finite capacity cut is at least

$$d(|V| - |S|) + |\{e \in E : e \cap S \neq \emptyset\}|$$

And this is at least $d|V|$ if (3) holds.

0-1 Matrices

Theorem

Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be two sets of non-negative integers where $b_1 \geq \cdots \geq b_n$. Then there is an $m \times n$ $0-1$ matrix $M = (M_{i,j})$ satisfying

$$\sum_{i=1}^{m} M_{i,j} = b_j, j \in [n] \quad \text{and} \quad \sum_{j=1}^{n} M_{i,j} \leq a_i, i \in [m] \tag{4}$$

iff

$$\sum_{j=1}^{k} b_j \leq \sum_{i \in A_k} a_i + k(m - |A_k|), \quad k = 0, \ldots, n-1, \tag{5}$$

where $A_k = \{i : a_i < k\}$

Proof Suppose first that the matrix M exists. Fix k and observe that the number of 1's in the first k rows is $b_1 + \cdots + b_k$.

On the other hand the number of 1's in the whole matrix is at least $\sum_{i \in A_k} a_i + k(m - |A_k|)$ and so (5) holds.

Now suppose that (5) holds. Define a network N as follows; the vertices are s, t, R, C where $R = \{r_1, \ldots, r_n\}$, $C = \{c_1, \ldots, c_n\}$.

There is an edge of capacity b_i from s to $r_i, i \in [n]$; an edge of capacity a_j from c_j to $t, j \in [n]$; an edge of capacity 1 from r_i to b_j.

Then matrix M exists if there is a flow f of value $b_1 + \cdots + b_n$ from s to t. It is defined by $M_{i,j} = f(r_i, c_j)$.

Let $X : \bar{X}$ be an $s - t$ cut and let $S = X \cap R$, $T = X \cap C$ where $|S| = k$. The capacity of $X : \bar{X}$ is

$$\begin{aligned}
&\sum_{i \notin S} b_i + \sum_{j \in T} a_j + |S|(n - |T|) \\
&\geq \sum_{i=k+1}^{n} b_i + \sum_{j \in A_k} a_j + k(n - |A_k|) \\
&= \sum_{i=1}^{n} b_i + \left(\sum_{j \in A_k} a_j + k(n - |A_k|) - \sum_{i=1}^{k} b_i \right) \\
&\geq \sum_{i=1}^{n} b_i,
\end{aligned}$$

as we have assume that (5) holds. Applying the Max-Flow Min-Cut theorem, we see that there is a flow of value $b_1 + \cdots + b_n$.