LINEEAR ALGEBRAIC METHODS

Oddtown

In order to cut down the number of committees a town of n people has instituted the following rules:

(a) Each club shall have an odd number of members.
(b) Each pair of clubs shall share an even number of members.

Theorem

With these rules, there are at most n clubs.

Proof Suppose that the clubs are $C_1, C_2, \ldots, C_m \subseteq [n]$.

Let $\mathbf{v}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$ denote the incidence vector of C_i for $1 \leq i \leq m$ i.e. $v_{i,j} = 1$ iff $j \in C_i$. We treat these vectors as being over the two element field $\mathbb{F}_2$.

We claim that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent and the theorem will follow.

The rules imply that (i) $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ and (ii) $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $1 \leq i \neq j \leq m$.
(Remember that we are working over $\mathbb{F}_2$.)

Suppose then that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = 0.$$

We show that $c_1 = c_2 = \cdots = c_m = 0$.

Indeed, we have

$$\begin{aligned} 0 &= \mathbf{v}_j \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m) \\ &= c_1\mathbf{v}_1 \cdot \mathbf{v}_j + c_2\mathbf{v}_2 \cdot \mathbf{v}_j + \cdots + c_m\mathbf{v}_m \cdot \mathbf{v}_j \\ &= c_j, \end{aligned}$$

for $j = 1, 2, \ldots, m$. □

Point sets in $\mathbb{R}^n$ with only two distances

Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\} \subseteq \mathbb{R}^n$.

Suppose that the pair-wise distance between elements of A only take two values. How large can A be?

Denote the maximum size of A by $m(n)$.

Theorem

$$\frac{n(n+1)}{2} \leq m(n) \leq \frac{(n+1)(n+4)}{2}.$$

Proof For the lower bound we let $A = \left\{\mathbf{e}_i + \mathbf{e}_j : 1 \leq i < j \leq n\right\}$ where $\mathbf{e}_i$ is the ith coordinate vector.

Point sets in $\mathbb{R}^n$ with only two distances

There are only two distances between elements of A viz. $2^{1/2}$ and 2.

$|A| = \frac{n(n-1)}{2}$, but it lies in the $(n-1)$-dimensional space $x_1 + x_2 + \cdots + x_n = 2$.

For the upper bound, assume that the two distances in A are d_1, d_2. Then consider the multivariate polynomial with $2n$ variables,

$$F(\mathbf{x}, \mathbf{y}) = (||\mathbf{x} - \mathbf{y}||^2 - d_1^2)(||\mathbf{x} - \mathbf{y}||^2 - d_2^2).$$

Thus our two-distance condition can be expressed:

$$F(\mathbf{a}_i, \mathbf{a}_j) = \begin{cases} (d_1 d_2)^2 & i = j \\ 0 & i \neq j \end{cases}$$

Next let

$$f_i(\mathbf{x}) = F(\mathbf{x}, \mathbf{a}_i) \text{ for } i = 1, 2, \ldots, m.$$

We claim that $f_1, f_2, \ldots, f_m$ are linearly independent over $\mathbb{R}$. Suppose that for some $\lambda_1, \lambda_2, \ldots, \lambda_m$

$$\lambda_1 f_1(\mathbf{x}) + \lambda_2 f_2(\mathbf{x}) + \cdots + \lambda_m f_m(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

But if $\mathbf{x} = \mathbf{a}_j$ then $f_i(\mathbf{x}) = 0$ for $i \neq j$ and $f_j(\mathbf{x}) = (d_1 d_2)^2 \neq 0$.

It follows that $\lambda_j f_j(\mathbf{a}_j) = 0$ and the independence claim follows.

Point sets in $\mathbb{R}^n$ with only two distances

On the other hand, all polynomials f_i can be expressed as linear combinations of the following:

$$\left(\sum_{k=1}^{n} x_k^2\right)^2, \left(\sum_{k=1}^{n} x_k^2\right) x_j, x_i x_j, x_i, 1.$$

The number of polynomials listed is

$$1 + n + n(n+1)/2 + n + 1 = (n+1)(n+4)/2.$$

Thus the f_i belong to a space of dimension at most $(n+1)(n+4)/2$. □

Decomposing K_n into bipartite subgraphs

Here we show

Theorem

If $G_k, k = 1, 2, \ldots, m$ is a collection of complete bipartite graphs with vertex partitions A_k, B_k, such that every edge of K_n is in exactly one subgraph, then $m \geq n - 1$. (Note that $A_k \cap B_k = \emptyset$ here.)

Proof This is tight since we can take $A_k = \{k\}, B_k = \{k + 1, \ldots, n\}$ for $k = 1, 2, \ldots, n - 1$.

Define $n \times n$ matrices M_k where $M_k(i, j) = 1$ if $i \in A_k, j \in B_k$ and $M_k(i, j) = 0$ otherwise.

Let $S = M_1 + M_2 + \cdots + M_m$. Then $S + S^T = J_n - I_n$ where I_n is the identity matrix and J_n is the all ones matrix.

Decomposing K_n into bipartite subgraphs

We show next that $rank(S) \geq n-1$ and then the theorem follows from

$$rank(S) \leq rank(M_1) + rank(M_2) + \cdots + rank(M_m) \leq m.$$

Suppose then that $rank(S) \leq n-2$ so that there exists a non-zero solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ to the system of equations

$$S\mathbf{x} = 0, \sum_{i=1}^{n} x_i = 0.$$

But then, $J_n\mathbf{x} = 0$ and $S^T\mathbf{x} = -\mathbf{x}$ and $-|\mathbf{x}|^2 = -\mathbf{x}^T S^T \mathbf{x} = 0$, contradiction. □

Nonuniform Fisher Inequality

Theorem

Let $C_1, C_2, \ldots, C_m$ be distinct subsets of $[n]$ such that for every $i \neq j$ we have $|C_i \cap C_j| = s$ where $1 \leq s < n$. Then $m \leq n$.

Proof If $|C_1| = s$ then $C_i \supset C_1, i = 2, 3, \ldots, m$ and the sets $C_i \setminus C_1$ are pairwise disjoint for $i \geq 2$.

It follows in this case that $m \leq 1 + n - s \leq n$.

Assume from now on that $c_i = |C_i| - s > 0$ for $i \in [m]$.

Nonuniform Fisher Inequality

Let M be the $m \times n$ $0/1$ matrix where $M(i,j) = 1$ iff $j \in C_i$.

Let

$$A = MM^T = sJ + D$$

where J is the $m \times m$ all 1's matrix and D is the diagonal matrix, where $D(i,i) = c_i$.

We show that A and hence M has rank m, implying that $m \leq n$ as claimed.

We will in fact show that $\mathbf{x}^T A\mathbf{x} > 0$ for all $0 \neq \mathbf{x} \in \mathbb{R}^m$. This means that $A\mathbf{x} \neq 0$ when $\mathbf{x} \neq 0$.

Nonuniform Fisher Inequality

If $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ then

$$\mathbf{x}^T A\mathbf{x} = s(x_1 + x_2 + \cdots + x_m)^2 + \sum_{i=1}^{m} c_i x_i^2 > 0.$$

$\square$

Lighting problem

Let $G = (V, E)$ be an arbitrary graph. Suppose that each vertex contains a light bulb and $\ell(v) = 1$ indicates that the light bulb on v is on and $\ell(v) = 0$ indicates that it is off.

Suppose that for $v \in V$, the transformation $T(v)$ flips the values at v and all of its neighbors. I.e. $T(v)$ switches on a neighboring light bulb if it is off and turns it off if it is on.

Suppose that initially, $\ell(v) = 0$ for all $v \in V$, i.e. all light bulbs are off. We show that there exists a set $S \subseteq V$ such that applying $T(v), v \in S$ in any order makes $\ell(v) = 1$ for $v \in V$.

Lighting problem

Observe first that applying $T(v)$ and then $T(w)$ achieves the same effect as applying $T(w)$ and then $T(v)$ i.e. the order of application of the transformations does not matter.
(The value of $\ell(u)$ is flipped by the two transformations iff it is adjacent to exactly one of $\{v, w\}$.)

Let A be the 0-1 adjacency matrix of G i.e. let $A(v, w) = 1$ iff $w \in N(v)$. In addition put $A(v, v) = 1$ for $v \in V$.

The set of transformations corresponding to S will turn on all of the lights iff $A\mathbf{1}_S = \mathbf{1}_V$ where $\mathbf{1}_S$ is the 0-1 vector indexed by V such that there is a 1 in component v iff $v \in S$.

Lighting problem

Our claim amounts to saying that there exists S such that $A\mathbf{1}_S = \mathbf{1}_V$ where calculations are done in the binary field.

If there is no such $\mathbf{1}_S$ then basic linear algebra theory tells us that there exists x such that $x^T A = 0$ and $x^T 1_V \neq 0$.

Since A is symmetric, this means that $Ax = 0$ as well. Let $x = 1_S$. Then S has the following properties:

(a) $|S \cap N(v)|$ is odd for all $v \in V$. This is a consequence of $Ax = 0$.

(b) $|S|$ is odd. This is a consequence of $x^T 1_V \neq 0$.

Lighting problem

Now consider the sub-graph of G induced by S.

Every vertex has odd degree by (a). But in any graph, the number of odd vertices is even. Contradiction.