SOME EXTREMAL PROBLEMS

Let $\mathcal{P}_n = \{A : A \subseteq [n]\}$ denote the *power set* of $[n]$.

$\mathcal{A} \subseteq \mathcal{P}_n$ is a *Sperner* family if $A, B \in \mathcal{A}$ implies that $A \not\subseteq B$ and $B \not\subseteq A$

Theorem

If $\mathcal{A} \subseteq \mathcal{P}_n$ is a Sperner family $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof We will show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1. \tag{1}$$

Now $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all k and so

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Proof of (1)**:** Let π be a random permutation of $[n]$.

For a set $A \in \mathcal{A}$ let $\mathcal{E}_A$ be the event

$$\{\pi(1), \pi(2), \ldots, \pi(|A|)\} = A.$$

If $A, B \in \mathcal{A}$ then the events $\mathcal{E}_A, \mathcal{E}_B$ are disjoint.

So

$$\sum_{A \in \mathcal{A}} \mathbf{Pr}(\mathcal{E}_A) \leq 1.$$

On the other hand, if $A \in \mathcal{A}$ then

$$\mathbf{Pr}(\mathcal{E}_A) = \frac{|A|!(n-|A|)!}{n!} = \frac{1}{\binom{n}{|A|}}$$

and (1) follows.

The set of all sets of size $\lfloor n/2 \rfloor$ is a Sperner family and so the bound in the above theorem is best possible.

Inequality (1) can be generalised as follows: Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1} \in \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$.

Theorem

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$

Proof Let π be a random permutation of $[n]$.

Let $\mathcal{E}(A)$ be the event $\{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\}$.

Let

$$Z_i = \begin{cases} 1 & \mathcal{E}(A_i) \text{ occurs.} \\ 0 & \text{otherwise.} \end{cases}$$

and let $Z = \sum_i Z_i$ be the number of events $\mathcal{E}(A_i)$ that occur.

Now our family is such that $Z \leq s$ for all π and so

$$E(Z) = \sum_i E(Z_i) = \sum_i \mathbf{Pr}(\mathcal{E}(A_i)) \leq s.$$

On the other hand, $A \in \mathcal{A}$ implies that $\mathbf{Pr}(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}}$ and the required inequality follows. $\square$

Intersecting Families

A family $\mathcal{A} \subseteq \mathcal{P}_n$ is an *intersecting* family if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$.

Theorem

If $\mathcal{A}$ is an intersecting family then $|\mathcal{A}| \leq 2^{n-1}$.

Proof Pair up each $A \in \mathcal{P}_n$ with its complement $A^c = [n] \setminus A$. This gives us 2^{n-1} pairs altogether. Since $\mathcal{A}$ is intersecting it can contain at most one member of each pair. □

If $\mathcal{A} = \{A \subseteq [n] : 1 \in A\}$ then $\mathcal{A}$ is intersecting and $|\mathcal{A}| = 2^{n-1}$ and so the above theorem is best possible.

Theorem

If $\mathcal{A}$ is an intersecting family and $A \in \mathcal{A}$ implies that $|A| = k \leq \lfloor n/2 \rfloor$ then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Proof If π is a permutation of $[n]$ and $A \subseteq [n]$ let

$$\theta(\pi, A) = \begin{cases} 1 & \exists s: \ \{\pi(s), \pi(s+1), \ldots, \pi(s+k-1)\} = A \\ 0 & \textit{otherwise} \end{cases}$$

where $\pi(i) = \pi(i-n)$ if $i > n$.

We will show that for any permutation π,

$$\sum_{A \in \mathcal{A}} \theta(\pi, A) \leq k. \tag{2}$$

Assume (2). We first observe that if π is a random permutation then

$$\mathbf{E}(\theta(\pi, A)) = n\frac{k!(n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

and so, from (2),

$$k \geq \mathbf{E}(\sum_{A\in\mathcal{A}} \theta(\pi, A)) = \sum_{A\in\mathcal{A}} \frac{k}{\binom{n-1}{|A|-1}}$$

Hence

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Assume w.l.o.g. that π is the identity permutation.

Let $A_t = \{t, t+1, \ldots, t+k-1\}$ and suppose that $A_s \in \mathcal{A}$.

All of the other sets A_t that intersect A_s can be partitioned into pairs A_{s-i}, A_{s+k-i}, $1 \leq i \leq k-1$ and the members of each pair are disjoint. Thus $\mathcal{A}$ can contain at most one from each pair. This verifies (2).

Kraft's Inequality

Let $x_1, x_2, \ldots, x_m$ be a collection of sequences over an alphabet Σ of size s. Let x_i have length n_i and let $n = \max\{n_1, n_2, \ldots, n_m\}$.

Assume next that no sequence is a prefix of any other sequence: Sequence $x_i = a_1 a_2 \cdots a_{n_i}$ is a prefix of $x_j = b_1 b_2 \cdots b_{n_j}$ if $a_i = b_i$ for $i = 1, 2, \ldots, n_i$.

Theorem

$$\sum_{i=1}^{m} r^{-n_i} \leq 1.$$

Proof: Let x be a random sequence of length n. Let $\mathcal{E}_i$ be the event x_i is a prefix of x. Then

(a) $\mathbf{Pr}(\mathcal{E}_i) = r^{-n_i}$.

(b) The event $\mathcal{E}_i$, $i = 1, 2, \ldots, m$ are disjoint.
(If $\mathcal{E}_i$ and $\mathcal{E}_j$ both occur and $n_i \leq n_j$ then x_i is a prefix of x_j.

Property (b) implies that

$$\mathbf{Pr}\left(\bigcup_{i=1}^{m} \mathcal{E}_i\right) = \mathbf{Pr}(\mathcal{E}_1) + \mathbf{Pr}(\mathcal{E}_2) + \cdots + \mathbf{Pr}(\mathcal{E}_m) \leq 1.$$

The theorem now follows from Property (a). □

The trace of a set system

Let X be a set and suppose that $\mathcal{F} \subseteq 2^X$.

For $Y \subseteq X$ we let $\mathcal{F} \cap Y = \{F \cap Y : F \in \mathcal{F}\}$.

Then for positive integer k we let

$$f_{\mathcal{F}}(k) = \max \left\{ |\mathcal{F} \cap Y| : Y \in \binom{X}{k} \right\}.$$

We define the trace number of the system $\mathcal{F}$ by

$$tr(\mathcal{F}) = \max \{ m : f_{\mathcal{F}}(m) = 2^m \}.$$

Theorem

Suppose that $|X| = n$ *and* $\mathcal{F} \subseteq 2^X$ *and* $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$. *Then* $tr(\mathcal{F}) \geq k$.

Proof For $x \in X$ set

$$\mathcal{F}_x = \mathcal{F} \cap (X \setminus \{x\}) = \{A \setminus \{x\} : A \in \mathcal{F}\}.$$

Let $\phi_x : \mathcal{F} \to \mathcal{F}_x$ be given by $\phi_x(A) = A \setminus \{x\}$.

ϕ_x is onto and if $|\phi^{-1}(B)| \geq 2$ then $\phi^{-1}(B) = \{B, B \cup \{x\}\}$ and $x \notin B$.

Let

$$\mathcal{A}_x = \{A \in \mathcal{F} : x \in A, A \setminus \{x\} \in \mathcal{F}\}.$$

$$\mathcal{B}_x = \{B \in \mathcal{F} : x \notin B, B \cup \{x\} \in \mathcal{F}\}.$$

Then

$$|\mathcal{F}| - |\mathcal{F}_x| = |\mathcal{A}_x| = |\mathcal{B}_x|. \tag{3}$$

Note that if $tr(\mathcal{B}_x) \geq k-1$ then $tr(\mathcal{F}) \geq k$. Indeed, suppose $\mathcal{B}_x \cap Y = 2^Y$ where $|Y| = k-1$. Set $Z = Y \cup \{x\}$. Then

$$\mathcal{F} \cap Z \supset (\mathcal{A}_x \cup \mathcal{B}_x) \cap Z = 2^Z.$$

Because if $x \in U \subset Z$ then $U \setminus \{x\} = B \cap Y = B \cap Z$ for some $B \in \mathcal{B}_x$ by assumption. So $U = A \cap Z$ where $A = B \cup \{x\} \in \mathcal{A}_x$.

To complete the proof we use induction on $n+k$. For $n+k=1$ there is nothing to prove.

Suppose that $n+k \geq 2$ and the result is true for smaller values of $n+k$.

Let $x \in X$. If $|\mathcal{F}_x| > \sum_{j=0}^{k-1} \binom{n-1}{j}$ then $tr(\mathcal{F}_x) \geq k$ by induction and so $tr(\mathcal{F}_x) \geq k$. Otherwise, by (3),

$$|\mathcal{B}_x| = |\mathcal{F}| - |\mathcal{F}_x| > \sum_{j=0}^{k-1} \binom{n}{j} - \sum_{j=0}^{k-1} \binom{n-1}{j}$$
$$= \sum_{j=1}^{k-1} \binom{n-1}{j-1} = \sum_{j=0}^{k-2} \binom{n-1}{j}.$$

Hence, by induction, $tr(\mathcal{B}_x) \geq k-1$ and so $tr(\mathcal{B}) \geq k$. □

Corollary

If $\mathcal{F}$ is a family of subsets of an infinite set S then either $f_{\mathcal{F}}(k) = 2^k$ for every k or else there exists ℓ such that $f_{\mathcal{F}}(n) \leq n^\ell$ for every $n \geq \ell$.

Proof Suppose that $f_{\mathcal{F}}(k) \neq 2^\ell$ for some ℓ. Then by the theorem,

$$f_{\mathcal{F}}(n) \leq \sum_{j=0}^{\ell-1} \binom{n}{j} \leq n^\ell \text{ for } n > \ell.$$

□

Sunflowers

A sunflower of size r is a family of sets $A_1, A_2, \ldots, A_r$ such that every element that belongs to more than one of the sets belongs to all of them.

Let $f(k, r)$ be the maximum size of a family of k-sets without a sunflower of size r.

Theorem

$f(k, r) \leq (r-1)^k k!$.

Proof Let $\mathcal{F}$ be a family of k-sets without a sunflower of size r. Let $A_1, A_2, \ldots, A_t$ be a maximum subfamily of pairwise disjoint subsets in $\mathcal{F}$.

Since a family of pairwise disjoint is a sunflower, we must have $t < r$.

Now let $A = \bigcup_{i=1}^{t} A_i$. For every $a \in A$ consider the family $\mathcal{F}_a = \{S \setminus \{a\} : S \in \mathcal{F}, a \in S\}$.

Now the size of A is at most $(r-1)k$.

The size of each $\mathcal{F}_a$ is at most $f(k-1, r)$. This is because a sunflower in $\mathcal{F}_a$ is a sunflower in $\mathcal{F}$.

So,

$$f(k,r) \leq (r-1)k \times f(k-1,r) \leq (r-1)k \times (r-1)^{k-1}(k-1)!,$$

by induction. □

Distinct Distances

Suppose that $X_1, X_2, \ldots, X_n$ are n points in the plane. We put bounds on the number of distinct distances among $|X_i X_j|$.

Let $f(n)$ denote the minimum among all sets of n points.

Lower bound: $f(n) \geq (n - 3/4)^{1/2} - 1/2$.

Assume that X_1 is a vertex of the least (in y value) convex polygon contained in the points. Let K be the number of distinct values among $\{|X_1 X_i| : i \geq 2\}$.

If N is the maximum number of times the same distance occurs then $KN \geq n - 1$.

If r is a distance that occurs N times then there are N points on the circle with center X_1 and radius r. They all lie on a semi-circle.

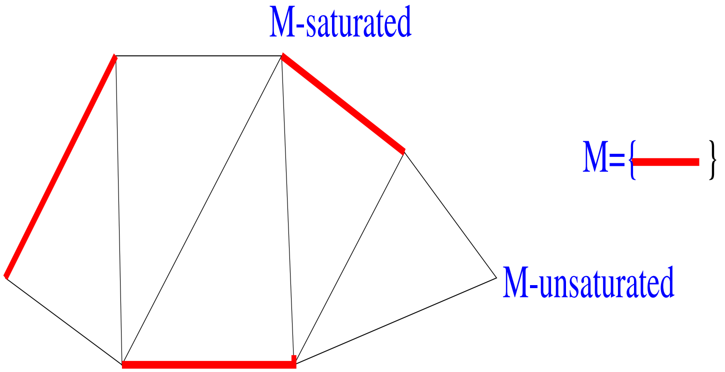
Going round the circle, let these points be $Q_1, Q_2, \ldots, Q_N$. Then $|Q_1 Q_2| < |Q_1 Q_3| \cdots < |Q_1 Q_N|$.

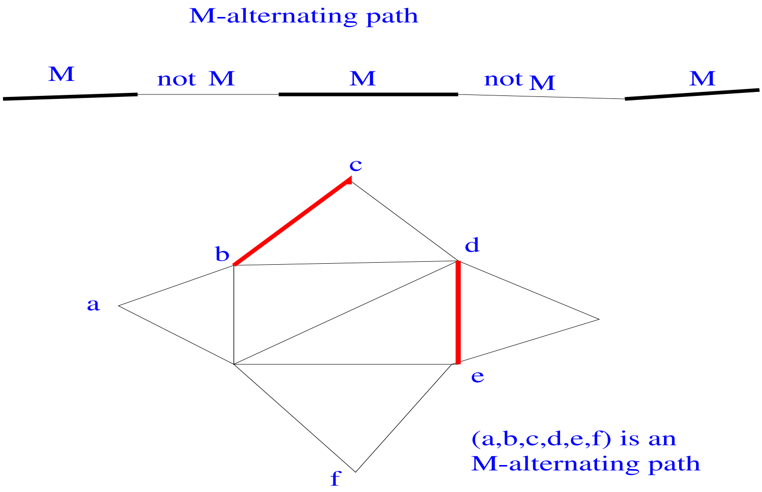
Thus $f(n) \geq \max\{(n-1)/N, N-1\}$. $N(N-1)$ minimises this lower bound and gives us what we claim.

Upper bound: we consider the integer points $\{(x, y)\}$ where $0 \leq x, y \leq n^{1/2}$. These have distance of the form $(u^2 + v^2)^{1/2}$ and $cn/\log^{1/2} n$ is a bound on the number of integers of the form $0 \leq u^2 + v^2 \leq 2n$.

Matchings

A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





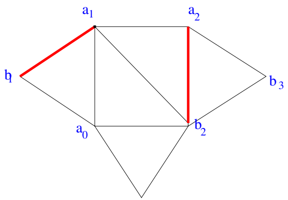
An *M*-alternating path joining 2 *M*-unsaturated vertices is called an *M*-augmenting path.

M is a *maximum* matching of G if no matching M' has more edges.

Theorem

M is a maximum matching iff M admits no M-augmenting paths.

Proof Suppose M has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \le i \le k+1$ and $f_i = (b_i, a_i) \in M$, $1 \le i \le k$.



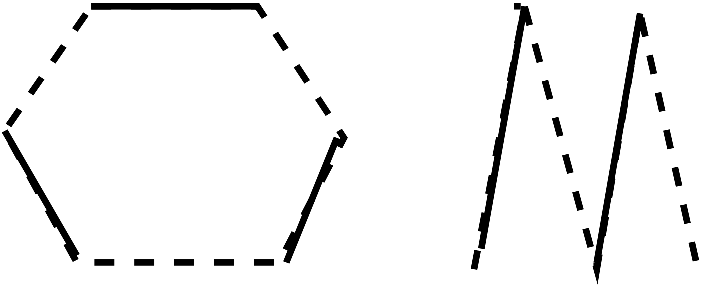
$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$

- $|M'| = |M| + 1$.
- M' is a matching

For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1. $d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$

So if M has an augmenting path it is not maximum.

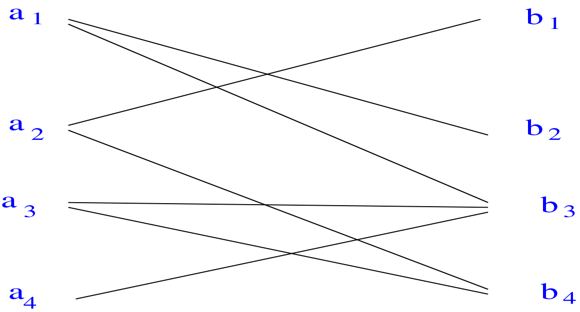
Suppose M is not a maximum matching and $|M'| > |M|$.
Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.
Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



(a) (b)

Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.
For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.



$N(\{a_2, a_3\}) = \{b_1 b_3 b_4\}$

Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Systems of Distinct Representatives

Let $S_1, S_2, \ldots, S_m$ be arbitrary sets. A set $s_1, s_2, \ldots, s_m$ of m disitinct elements is a system of distinct representatives if $s_i \in S_i$ for $i = 1, 2, \ldots, m$.

For example $\{1, 2, 4\}$ is a system of distinct representatives for $\{1, 2, 3\}, \{2, 5, 6\}, \{2, 4, 5\}$.

Now define the bipartite graph G with vertex bipartition $[m], S$ where $S = \bigcup_{i=1}^{m} S_i$ and an edge (i, s) iff $s \in S_i$.

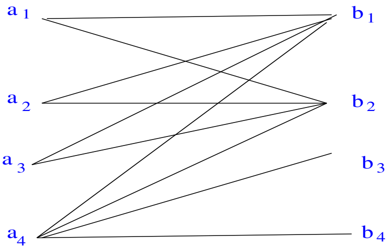
Then $S_1, S_2, \ldots, S_m$ has a system of distinct representatives iff G has a matching of size m.

Hall's Theorem

Theorem

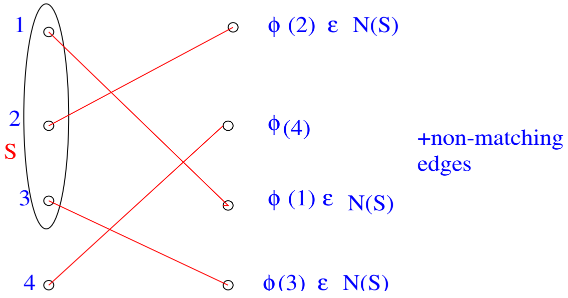
G contains a matching of size $|A|$ iff

$$|N(S)| \geq |S| \qquad \forall S \subseteq A. \tag{4}$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

Only if: Suppose $M = \{(a, \phi(a)) : a \in A\}$ saturates A.



$$\begin{aligned} |N(S)| &\geq |\{\phi(s) : s \in S\}| \\ &= |S| \end{aligned}$$

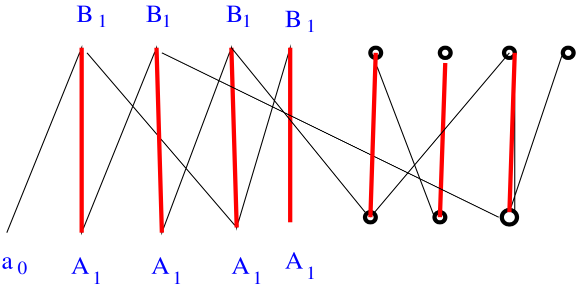
and so (4) holds.

If: Let $M = \{(a, \phi(a)) : a \in A'\}$ ($A' \subseteq A$) is a maximum matching. Suppose $a_0 \in A$ is M-unsaturated. We show that (4) fails.

Let

$A_1 = \{a \in A$: such that a is reachable from a_0 by an M-alternating path.$\}$

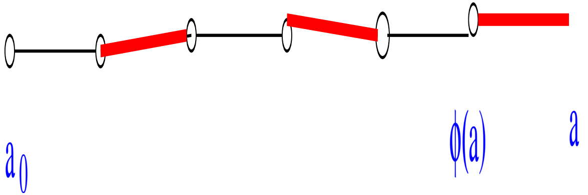
$B_1 = \{b \in B$: such that b is reachable from a_0 by an M-alternating path.$\}$



No A 1 - B\B 1

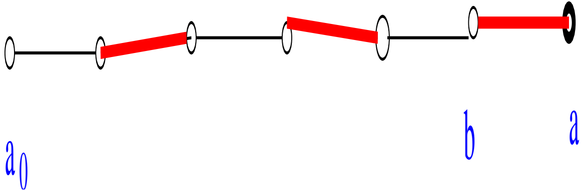
edges

- B_1 is M-saturated else there exists an M-augmenting path.
- If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.



- If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.

So $|B_1| = |A_1| - 1$. • $N(A_1) \subseteq B_1$



So $|N(A_1)| = |A_1| - 1$ and (4) fails to hold.

Marriage Theorem

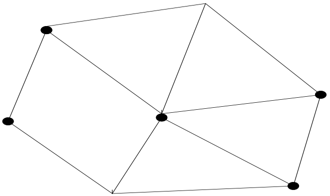
Theorem

Suppose $G = (A \cup B, E)$ *is* k*-regular. (*$k \geq 1$*) i.e.* $d_G(v) = k$ *for all* $v \in A \cup B$*. Then* G *has a perfect matching.*

Proof $k|A| = |E| = k|B|$ and so $|A| = |B|$.
Suppose $S \subseteq A$. Let m be the number of edges incident with S. Then $k|S| = m \leq k|N(S)|$. So (4) holds and there is a matching of size $|A|$ i.e. a perfect matching.

Edge Covers

A set of vertices $X \subseteq V$ is a *covering* of $G = (V, E)$ if every edge of E contains at least one endpoint in X.



$\{\bullet\}$ is a covering

Lemma

If X is a covering and M is a matching then $|X| \geq |M|$.

Proof Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. □

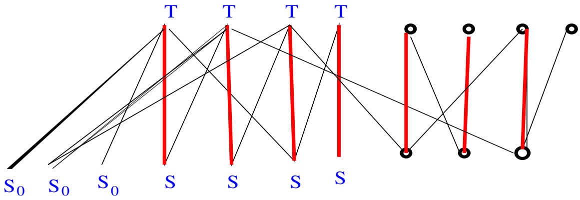
Konig's Theorem

Let $\mu(G)$ be the maximum size of a matching.
Let $\beta(G)$ be the minimum size of a covering.
Then $\mu(G) \leq \beta(G)$.

Theorem

If G is bipartite *then $\mu(G) = \beta(G)$.*

Proof Let M be a maximum matching.
Let S_0 be the M-unsaturated vertices of A.
Let $S \supseteq S_0$ be the A-vertices which are reachable from S_0 by M-alternating paths.
Let T be the M-neighbours of $S \setminus S_0$.



Let $X = (A \setminus S) \cup T$.

• $|X| = |M|$.

$|T| = |S \setminus S_0|$. The remaining edges of M cover $A \setminus S$ exactly once.

• X is a cover.

There are no edges (x, y) where $x \in S$ and $y \in B \setminus T$. Otherwise, since y is M-saturated (no M-augmenting paths) the M-neighbtour of y would have to be in S, contradicting $y \notin T$. $\square$