COMBINATORIAL GAMES

Game 1

Start with n chips. Players A,B alternately take 1,2,3 or 4 chips until there are none left. The winner is the person who takes the last chip:

Example

	A	B	A	B	A	
$n = 10$	3	2	4	1		B wins
$n = 11$	1	2	3	4	1	A wins

What is the optimal strategy for playing this game?

Game 2

Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') where $0 \leq m' < m$ and $0 \leq n' < n$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 2a Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') or to $(m - a, n - a)$ where $0 \leq m' < m$ and $0 \leq n' < n$ and $0 \leq a \leq \min\{m, n\}$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 3

W is a set of words. A and B alternately remove words $w_1, w_2, \ldots,$ from W. The rule is that the first letter of w_{i+1} must be the same as the last letter of w_i. The player who makes the last legal move wins.

Example

$W = \{England, France, Germany, Russia, Bulgaria, \ldots\}$

What is the optimal strategy for this game?

Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$.
(x, y) is an arc of D iff one can move from position x to position y.

We assume that the digraph is finite and that it is **acyclic** i.e. there are no directed cycles.

The game starts with a token on vertex x_0 say, and players alternately move the token to $x_1, x_2, \ldots,$ where $x_{i+1} \in N^+(x_i)$, the set of out-neighbours of x_i. The game ends when the token is on a **sink** i.e. a vertex of out-degree zero. The last player to move is the winner.

Abstraction

Example 1: $V(D) = \{0, 1, \ldots, n\}$ and $(x, y) \in A$ iff $x - y \in \{1, 2, 3, 4\}$.

Example 2: $V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and $(x, y) \in N^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$.

Example 2a: $V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and $(x, y) \in N^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$ or $x - x' = y - y' > 0$.

Example 3: $V(D) = \{(W', w) : \ W' \subseteq W \setminus \{w\}\}$. w is the last word used and W' is the remaining set of unused words. $(X', w') \in N^+((X, w))$ iff $w' \in X$ and w' begins with the last letter of w. Also, there is an arc from $(W, \cdot)$ to $(W \setminus \{w\}, w)$ for all w, corresponding to the games start.

Abstraction

We will first argue that such a game must eventually end.

A **topological numbering** of digraph $D = (X, A)$ is a map $f : X \to [n]$, $n = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

Theorem

A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.

Proof Suppose first that D has a topological numbering. We show that it is acyclic.

Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.

Suppose now that D is acyclic. We first argue that D has at least one sink.

Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in D. We claim that x_k is a sink.

If D contains an arc (x_k, y) then either $y = x_i, 1 \leq i \leq k-1$ and this means that D contains the cycle $(x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then (P, y) is a longer simple path than P, contradiction.

Abstraction

We can now prove by induction on n that there is at least one topological numbering.

If $n = 1$ and $X = \{x\}$ then $f(x) = 1$ defines a topological numbering.

Now asssume that $n > 1$. Let z be a sink of D and define $f(z) = n$. The digraph $D' = D - z$ is acyclic and by the induction hypothesis it admits a topological numbering, $f : X \setminus \{z\} \to [n - 1]$.

The function we have defined on X is a topological numbering. If $(x, y) \in A$ then either $x, y \neq z$ and then $f(x) < f(y)$ by our assumption on f, or $y = z$ and then $f(x) < n = f(z)$ ($x \neq z$ because z is a sink). □

Abstraction

The fact that D has a topological numbering implies that the game must end. Each move increases the f value of the current position by at least one and so after at most n moves a sink must be reached.

The positions of a game are partitioned into 2 sets:

- P-positions: The next player cannot win. The previous player can win regardless of the current player's strategy.
- N-positions: The next player has a strategy for winning the game.

Thus an N-position is a winning position for the next player and a P-position is a losing position for the next player.

The main problem is to determine N and P and what the strategy is for winning from an N-position.

Abstraction

Let the vertices of D be $x_1, x_2, \ldots, x_n$, in topological order.

Labelling procedure

1. $i \leftarrow n$, Label x_n with P. $N \leftarrow \emptyset,\ P \leftarrow \emptyset$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. Label x_i with N, if $N^+(x_i) \cap P \neq \emptyset$.
4. Label x_i with P, if $N^+(x_i) \subseteq N$.
5. goto 2.

The partition N, P satisfies

$$x \in N \textit{ iff } N^+(x) \cap P \neq \emptyset$$

To play from $x \in N$, move to $y \in N^+(x) \cap P$.

Abstraction

In Game 1, $P = \{5k : \ k \geq 0\}$.

In Game 2, $P = \{(x, x) : \ x \geq 0\}$.

Lemma

The partition into N, P satisfying $x \in N$ iff $N^+(x) \cap P \neq \emptyset$ is unique.

Proof If there were two partitions $N_i, P_i, \ i = 1, 2$, let x_i be the vertex of highest topological number which is not in $(N_1 \cap N_2) \cup (P_1 \cap P_2)$. Suppose that $x_i \in N_1 \setminus N_2$.

But then $x_i \in N_1$ implies $N^+(x_i) \cap P_1 \cap \{x_{i+1}, \ldots, x_n\} \neq \emptyset$ and $x_i \in P_2$ implies $N^+(x_i) \cap P_2 \cap \{x_{i+1}, \ldots, x_n\} = \emptyset$.

But $P_1 \cap \{x_{i+1}, \ldots, x_n\} = P_2 \cap \{x_{i+1}, \ldots, x_n\}$. □

Sums of games

Suppose that we have p games $G_1, G_2, \ldots, G_p$ with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, p$.
The sum $G_1 \oplus G_2 \oplus \cdots \oplus G_p$ of these games is played as follows. A position is a vector $(x_1, x_2, \ldots, x_p) \in X = X_1 \times X_2 \times \cdots \times X_p$. To make a move, a player chooses i such that x_i is not a sink of D_i and then replaces x_i by $y \in N_i^+(x_i)$. The game ends when each x_i is a sink of D_i for $i = 1, 2, \ldots, n$.

Knowing the partitions N_i, P_i for game $i = 1, 2, \ldots, p$ does not seem to be enough to determine how to play the sum of the games.

We need more information. This will be provided by the Sprague-Grundy Numbering

Example
Nim In a one pile game, we start with $a \geq 0$ chips and while there is a positive number x of chips, a move consists of deleting $y \leq x$ chips. In this game the N-positions are the positive integers and the unique P-position is 0.

In general, Nim consists of the sum of n single pile games starting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.

Example 2 is Nim with 2 piles.

Sprague-Grundy (SG) Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$mex(S) = \min\{x \geq 0 : \; x \notin S\}.$$

Now given an acyclic digraph $D = X, A$ with topological ordering $x_1, x_2, \ldots, x_n$ define g iteratively by

1. $i \leftarrow n$, $g(x_n) = 0$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. $g(x_i) = mex(\{g(x) : \; x \in N^+(x_i)\})$.
4. goto 2.

Lemma

$$x \in P \leftrightarrow g(x) = 0.$$

Proof Because

$$x \in N \textit{ iff } N^{+}(x) \cap P \neq \emptyset$$

all we have to show is that

$$g(x) > 0 \textit{ iff } \exists y \in N^{+}(y) \textit{ such that } g(y) = 0.$$

But this is immediate from $g(x) = mex(\{g(y) : \ y \in N^{+}(x)\})$ □

Sums of games

Another one pile subtraction game.

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma

$g(0) = 0$, $g(2k) = k - 1$ and $g(2k - 1) = k$ for $k \geq 1$.

Sums of games

Proof 0,2 are terminal postions and so $g(0) = g(2) = 0$. $g(1) = 1$ because the only position one can move to from 1 is 0. We prove the remainder by induction on k.

Assume that $k > 1$.

$$
\begin{aligned}
g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(2)\} \\
&= mex\{k-2, k-3, \ldots, 0\} \\
&= k-1. \\
g(2k-1) &= mex\{g(2k-3), g(2k-5), \ldots, g(1), g(0)\} \\
&= mex\{k-1, k-2, \ldots, 0\} \\
&= k.
\end{aligned}
$$

□

We now show how to compute the *SG* numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by

$$c_i = \begin{cases} 1 & \text{if } a_i \neq b_i \\ 0 & \text{if } a_i = b_i \end{cases}$$

for $i = 1, 2, \ldots, m$.

So $11 \oplus 5 = 14$.

Sums of games

Theorem

If g_i is the SG function for game G_i, $i = 1, 2, \ldots, p$ then the SG function g for the sum of the games $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ is defined by

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

where $x = (x_1, x_2, \ldots, x_p)$.

For example if in a game of Nim, the pile sizes are $x_1, x_2, \ldots, x_p$ then the *SG* value of the position is

$$x_1 \oplus x_2 \oplus \cdots \oplus x_p$$

Proof It is enough to show this for $p = 2$ and then use induction on p.

Write $G = H \oplus G_p$ where $H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1}$. Let h be the SG numbering for H. Then, if $y = (x_1, x_2, \ldots, x_{p-1})$,

$$\begin{aligned} g(x) &= h(y) \oplus g_p(x_p) \qquad \textit{assuming theorem for } p = 2 \\ &= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_{p-1}(x_{p-1}) \oplus g_p(x_p) \end{aligned}$$

by induction.

It is enough now to show, for $p = 2$, that

- A1 If $x \in X$ and $g(x) = b > a$ then there exists $x' \in N^+(x)$ such that $g(x') = a$.
- A2 If $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq g(x)$.
- A3 If $x \in X$ and $g(x) = 0$ and $x' \in N^+(x)$ then $g(x') \neq 0$

A1. Write $d = a \oplus b$. Then

$$a = d \oplus b = d \oplus g_1(x_1) \oplus g_2(x_2). \qquad (1)$$

Now suppose that we can show that either

$$(i)\ d \oplus g_1(x_1) < g_1(x_1) \text{ or } (ii)\ d \oplus g_2(x_2) < g_2(x_2) \text{ or both.} \qquad (2)$$

Assume that (i) holds.

Then since $g_1(x_1) = mex(N_1^+(x_1))$ there must exist $x_1' \in N_1^+(x_1)$ such that $g_1(x_1') = d \oplus g_1(x_1)$.

Then from (1) we have

$$a = g_1(x_1') \oplus g_2(x_2) = g(x_1', x_2).$$

Furthermore, $(x_1', x_2) \in N^+(x)$ and so we will have verified A1.

Sums of games

Let us verify (2).

Suppose that $2^{k-1} \leq d < 2^k$.

Then d has a 1 in position k and no higher.

Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$.

So either (i) $g_1(x_1)$ has a 1 in position k or (ii) $g_2(x_2)$ has a 1 in position k. Assume (i).

But then $d \oplus g_1(x_1) < g_1(x_1)$ since d “destroys” the kth bit of $g_1(x_1)$ and does not change any higher bit.

A2. Suppose without loss of generality that $g(x_1', x_2) = g(x_1, x_2)$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$ implies that $g_1(x_1') = g_1(x_1)$, contradition. □

A3. Suppose that $g_1(x_1) \oplus g_2(x_2) = 0$ and $g_1(x_1') \oplus g_2(x_2) = 0$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1) = g_1(x_1')$, contradicting $g_1(x_1) = mex\{g_1(x) : x \in N^+(x_1)\}$.

Sums of games

If we apply this theorem to the game of Nim then if the position x consists of piles of x_i chips for $i = 1, 2, \ldots, p$ then $g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_p$.

In our first example, $g(x) = x \mod 5$ and so for the sum of p such games we have

$$g(x_1, x_2, \ldots, x_p) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_p \mod 5).$$

A more complicated one pile game

Start with n chips. First player can remove up to $n-1$ chips.

In general, if the previous player took x chips, then the next player can take $y \leq x$ chips.

Thus a games position can be represented by (n, x) where n is the current size of the pile and x is the maximum number of chips that can be removed in this round.

Theorem

Suppose that the position is (n, x) where $n = m2^k$ and m is odd. Then,

(a) *This is an N-position if $x \geq 2^k$.*

(b) *This is a P-position if $m = 1$ and $x < n$.*

A more complicated one pile game

Proof For a non-negative integer $n = m2^k$, let $ones(n)$ denote the number of ones in the binary expansion of n and let $k = \rho(n)$ determine the position of the right-most one in this expansion.

We claim that the following strategy is a win for the player in a postion described in (a):

Remove $y = 2^k$ chips.

Suppose this player is A.

If $m = 1$ then $x \geq n$ and A wins.

A more complicated one pile game

Otherwise, after such a move the position is (n', y) where $\rho(n') > \rho(n)$.

Note first that $ones(n') = ones(n) - 1 > 0$ and $\rho(n') > k$. B cannot remove more than 2^k chips and so B cannot win at this point.

If B moves the position to (n'', x'') then $ones(n'') > ones(n')$ and furthermore, $x'' \geq 2^{\rho(n'')}$, since x'' must have a 1 in position $\rho(n'')$. ($\rho(n'')$ is the least significant bit of x''.)

Thus, by induction, A is in an N-position and wins the game.

To prove (b), note that after the first move, the position satisfies the conditions of (a). $\square$.

A General Subtraction Game

Let us next consider a generalisation of this game.

There are 2 players A and B and A goes first.

We have a non-decreasing function f from $\mathbf{N} \to \mathbf{N}$ where $\mathbf{N} = \{1, 2, \ldots\}$ which satisfies $f(x) \geq x$.

At the first move A takes any number less than h from the pile, where h is the size of the initial pile.

Then on a subsequent move, if a player takes x chips then the next player is constrained to take at most $f(x)$ chips.

Thus the previous analysis was for the game with $f(x) = x$.

A General Subtraction Game

There is a set $\mathcal{H} = \{H_1 = 1 < H_2 < \ldots\}$ of initial pile sizes for which the first player will lose, assuming that the second player plays optimally.

Also, if the initial pile size $h \notin \mathcal{H}$ then the first player has a winning strategy. It will turn out that the sequence satisfies the recurrence:

$$H_{j+1} = H_j + H_\ell \text{ where } H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}, \qquad \text{for } j \geq 0.$$

If $f(x) = x$ then $H_j = 2^{j-1}$.

We prove this inductively. It is true for $j = 1$.

$$\begin{aligned} H_{j+1} &= 2^{j-1} + \min_{i \le j}\{2^{i-1} :\ 2^{i-1} \ge 2^{j-1}\} \\ &= 2^{j-1} + 2^{j-1} \\ &= 2^j. \end{aligned}$$

A General Subtraction Game

If $f(x) = 2x$ then $\mathcal{H} = \{1, 2, 3, 5, 8, \ldots,\} = \{F_1, F_2, \ldots,\}$, the Fibonacci sequence.

We prove this inductively. It is true for $j = 1, 2$.

$$\begin{aligned} H_{j+1} &= F_j + \min_{i \le j}\{F_i :\ 2F_i \ge F_j\} \\ &= F_j + F_{j-1} \\ &= F_{j+1}. \end{aligned}$$

Recall that $F_j = F_{j-1} + F_{j-2}$ and

$$2F_{j-2} < F_{j-1} + F_{j-2} = F_j.$$

A General Subtraction Game

The key to the game is the following result.

Theorem

Every positive integer n can be uniquely written as the sum

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $1 \leq i < p$.

One simple consequence of the uniqueness of the decomposition is that

$$H_k \neq H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

for all k and sequences $j_1, j_2, \ldots, j_p$ where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, ..., p - 1$.

A General Subtraction Game

It follows that the integers n can be given unique “binary” representations by representing $n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$ by the 0-1 string with a 1 in posiitons $j_1, j_2, \ldots, j_p$ and 0 everywhere else.

Let $\rho_H(n) = p$ be the number of 1’s in the representation.

We call this the H-representation of n. This then leads to the following

Theorem

Suppose that the start position is $(n, *)$. Then,

(a) *This is an N-position if $n \notin \mathcal{H} = \{H_1, H_2, \ldots, \}$.*

(b) *This is a P-position if $n \in \mathcal{H}$.*

(a) The winning strategy is to delete a number of chips equal to H_{j_1} where j_1 is the index of the rightmost 1 in the H-representation of $n = H_{j_p} + \cdots + H_{j_1}$.

All we have to do is verify that this strategy is possible.

Note first that if A deletes H_{j_1} chips, then B cannot respond by deleting H_{j_2} chips, because $H_{j_2} > f(H_{j_1})$.

B is forced to delete $x \leq f(H_{j_1}) < H_{j_2}$ chips.

If $p = 2$ then $\rho_H(n - H_{j_1} - x) \geq 1 = \rho_H(n - H_{j_1})$.

If $p \geq 3$ and $y = H_{j_2} - x = H_{k_q} + \cdots + H_{k_1}$ then the H-representation of $n - H_{j_1} - x$ is

$$H_{j_p} + \cdots + H_{j_3} + H_{k_q} + \cdots + H_{k_1}.$$

Here we use the fact that $f(H_{k_q}) \leq f(y) \leq f(H_{j_2}) < H_{j_3}$.

And so in both cases $\rho_H(n - H_{j_2} - x) \geq \rho_H(n - H_{j_1})$ it is only A that can reduce ρ_H.

A General Subtraction Game

The next thing to check is that if A starts in $(n, *)$ then A can always delete H_{j_1} chips i.e. the positions (m, x) that A will face satisfy $f(x) \geq H_{k_1}$ where $m = H_{k_1} + H_{k_2} + \cdots + H_{k_q}$.

We do this by induction on the number of plays in the game so far.

It is true in the first move and suppose that it is true for (m, x) and that A removes H_{k_1} and B removes y where $y \leq \min\{m - H_{k_1}, f(H_{k_1})\} < H_{k_2}$. Now if $H_{k_2} - y = H_{\ell_r} + H_{\ell_{r-1}} + \cdots + H_{\ell_1}$ then

$$\begin{aligned} m - H_{k_1} - y &= H_{k_q} + \cdots + H_{k_3} + H_{k_2} - y \\ &= H_{k_q} + \cdots + H_{k_3} + H_{\ell_r} + H_{\ell_{r-1}} + \cdots + H_{\ell_1} \end{aligned}$$

and we need to argue that $H_{\ell_1} \leq f(y)$.

A General Subtraction Game

But if $f(y) < H_{\ell_1}$ then we have

$$\begin{aligned} H_{k_2} &= y + H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} \\ &= H_{a_1} + \cdots + H_{a_s} + H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} \end{aligned}$$

where $f(H_{a_s}) \leq f(y) < H_{\ell_1}$, which gives two distinct decompositons for H_{k_2}, contradiction.

Thus A can remove H_{ℓ_1} in the next round, as required.

A General Subtraction Game

(b) Assume that $n = H_k$. After A removes x chips we have

$$H_k - x = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

chips left.

All we have to show is that B can now remove H_{j_1} chips i.e. $H_{j_1} \leq f(x)$.

But if this is not the case then we argue as above that $H_k = H_{a_1} + \cdots + H_{a_s} + H_{j_1} + H_{j_2} + \cdots + H_{j_p}$, where $x = H_{a_1} + \cdots + H_{a_s}$ and $f(H_{j_1}) \leq f(x) < H_{j_1}$, which gives two distinct decompositons for H_k, contradiction.

Proof of the existence of a unique decomposition

We prove this by induction on n. If $n = 1$ then $n = H_1$ is the unique decomposition.
Going back to the defining recurrence we see that

$$H_{j+1} = H_j + H_\ell \leq 2H_j.$$

Existence
Assume that any $n < H_k$ can be represented as a sum of distinct H_{j_i}'s with $f(H_{j_i}) < H_{j_{i+1}}$ and suppose that $H_k \leq n < H_{k+1}$. $H_{k+1} \leq 2H_k$ implies that $n - H_k < H_k$.
It follows by induction that

$$n - H_k = H_{j_1} + \cdots + H_{j_p},$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, ..., p - 1$.

A General Subtraction Game

Assume to the contrary that $f(H_{j_p}) \geq H_k$.

Then for some $m \leq j_p$ we have

$$H_{k+1} = H_k + H_m \leq H_k + H_{j_p} \leq n,$$

contradicting the choice of n.

Uniqueness

We will first prove by induction on p that if $f(H_{j_i}) < H_{j_{i+1}}$ for $1 \leq i < p$ then

$$H_{j_1} + H_{j_2} + \cdots + H_{j_p} < H_{j_p+1}. \quad (3)$$

If $p = 2$ then we are saying that if $f(H_{j_1}) < H_{j_2}$ then $H_{j_1} + H_{j_2} < H_{j_2+1}$. But this follows directly from $H_{j_2+1} = H_{j_2} + H_m$ where $f(H_m) \geq H_{j_2}$ i.e. $H_m > H_{j_1}$.

So assume that (3) is true for $p \geq 2$. Now

$$H_{j_{p+1}+1} = H_{j_{p+1}} + H_m \text{ and } f(H_{j_p}) < H_{j_{p+1}}$$

implies that $m \geq j_p + 1$. Thus

$$\begin{aligned} H_{j_{p+1}+1} &\geq H_{j_{p+1}} + H_{j_p+1} \\ &> H_{j_{p+1}} + H_{j_p} + H_{j_{p-1}} + \cdots + H_{j_1} \end{aligned}$$

after applying induction to get the second inequality.

This completes the induction for (3).

A General Subtraction Game

Now assume by induction on k that $n < H_k$ has a unique decomposition. This is true for $k = 2$ and so now assume that $k \geq 2$ and $H_k \leq n < H_{k+1}$. Consider a decomposition

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}.$$

It follows from (3) that $j_p = k$. Indeed, $j_p \leq k$ since $n < H_{k+1}$ and if $j_p < k$ then $H_{j_1} + H_{j_2} + \cdots + H_{j_p} < H_{j_p+1} \leq H_k$, contradicting our choice of n. So H_k appears in every decomposition of n.

Now $H_{k+1} \leq 2H_k$ and $n < H_{k+1}$ implies $n - H_k < H_k$ and so, by induction, $n - H_k$ has a unique decompositon. But then if n had two distinct decompositions, H_k would appear in each, implying that $n - H_k$ also had two distinct decompositions, contradiction.

Note that although we know the optimal strategy for this game, we do not know the Sprague-grundy numbers and so we do not immediately get a solution to multi-pile versions.

This is Game 2a.

Theorem

The set of P-positions is $\mathcal{A} = ((a_i, b_i),\ i = 0, 1, 2, \ldots)$ where $a_i < b_i,\ i \neq 0$ can be generated as follows: $a_0 = b_0 = 0$ and

- *a_i is the smallest integer not appearing in $a_0, b_0, \ldots, a_{i-1}, b_{i-1}$*
- *$b_i = a_i + i$.*

The sequence $\mathcal{A}$ starts

0	0	1	2	3	5
4	7	6	10	8	13
9	15	11	18	12	20
14	23	16	26	17	28
19	31	21	34	22	36
24	39	25	41	27	44

Proof We first prove that each positive integer appears exactly once either as a_i or b_i.

We cannot have $a_i = a_j$ for $i < j$ because a_j is the smallest integer that has not previously appeared. Similarly, we cannot have $a_i < a_{i-1}$, else a_{i-1} was too large.

Since $b_i = a_i + i$ we see that both of the sequences $a_0, a_1, \ldots,$ and $b_0, b_1, \ldots,$ are monotone increasing.

Suppose then that $x = a_i = b_j$. Since $a_i < b_i < b_j$ for $i < j$, we must have $i > j$ here. But then a_i is not an integer that has not appeared before.

Thus each positive integer appears exactly once either as a_i or b_i.

Wythoff's Nim

Now suppose that $(a_i, b_i) \in \mathcal{A}$. We consider the possible positions we can move to and check that we cannot move to $\mathcal{A}$:

1. $(a_i - x, b_i) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $b_j = b_i$. Not possible.
2. $(a_i, b_i - x) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $a_j = a_i$. Not possible.
3. $(a_i - x, b_i - x) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $i = b_i - a_i = b_j - a_j = j$. Not possible.

Wythoff's Nim

Now suppose that $(c, d) \notin \mathcal{A}, c, d$. We see that we can move to a pair in $\mathcal{A}$.

1. $c = a_i$ and $d > b_i$.
   We can move to (a_i, b_i) by removing $d - b_i$ from the d pile.
2. $c = a_i$ and $d < b_i$.
   Let $j = d - c$. We can move to (a_j, b_j) by deleting $c - a_j = d - b_j$ from each pile.
3. $d = b_i$ and $c > a_i$.
   We can move to (a_i, b_i) by removing $c - a_i$ from the c pile.
4. $d = b_i$ and $c < a_i$ and we are not in Case 1 (with i replaced by i').
   Thus, $c = b_j$ for some $j < i$. We can move to (a_j, b_j) by removing $d - a_j$ from the d pile.

We have therefore verified that the sequence $\mathcal{A}$ does indeed define the set of P positions.

We can give the following description of the sequence $\mathcal{A}$.

Theorem

$$a_k = \lfloor \tfrac{k}{2}(1+\sqrt{5}) \rfloor \text{ and } b_k = \lfloor \tfrac{k}{2}(3+\sqrt{5}) \rfloor$$

for $k = 0, 1, 2, \ldots$.

Proof It will be enough to show that each non-negative integer appears exactly once in the sequence

$$(x_k, y_k) = \left(\lfloor \tfrac{k}{2}(1+\sqrt{5}) \rfloor, \lfloor \tfrac{k}{2}(3+\sqrt{5}) \rfloor\right) \qquad (*).$$

Given (*) we assume inductively that $(a_i, b_i) = (x_i, y_i)$ for $0 \leq i \leq k$. This is true for $k = 0$.

Using (*) we see that a_{k+1} appears in some pair x_j, y_j. We must have $j > k$ else a_{k+1} will appear in $a_0, \ldots, b_k$.

Now x_{k+1} is the smallest integer that that does not appear in $(x_0, \ldots, y_k) = (a_0, \ldots, b_k)$ and so $x_{k+1} = a_{k+1}$ and then $y_{k+1} = x_{k+1} + k = b_{k+1}$, completing the induction.

Proof of (*)

Fix an integer n and write

$$\alpha = \frac{1}{2}p(1+\sqrt{5}) - n \quad (4)$$

$$\beta = \frac{1}{2}q(3+\sqrt{5}) - n \quad (5)$$

where p, q are integers and

$$0 < \alpha < \frac{1}{2}p(1+\sqrt{5}) \quad (6)$$

$$0 < \beta < \frac{1}{2}q(3+\sqrt{5}) \quad (7)$$

Multiply (9) by $\frac{1}{2}(-1+\sqrt{5})$ and (10) by $\frac{1}{2}(3-\sqrt{5})$ and add to get

$$\frac{1}{2}\alpha(-1+\sqrt{5})+\frac{1}{2}\beta(3-\sqrt{5})=p+q-n=\ \mathit{integer}.$$

Multiply (6) by $\frac{1}{2}(-1+\sqrt{5})$ and (7) by $\frac{1}{2}(3-\sqrt{5})$ and add to get

$$0<\frac{1}{2}\alpha(-1+\sqrt{5})+\frac{1}{2}\beta(3-\sqrt{5})<2.$$

We see therefore that

$$\frac{1}{2}\alpha(-1+\sqrt{5})+\frac{1}{2}\beta(3-\sqrt{5})=p+q-n=1. \qquad (8)$$

Although $\alpha=\beta=1$ satisfies (8) this can be rejected by observing that (9) would then imply that $n+1=p(1+\sqrt{5})$.

Wythoff's Nim

Thus either (i) $\alpha < 1, \beta > 1$ or (ii) $\alpha > 1, \beta < 1$.

In case (i) we have from (9) that $n = \lfloor p(1+\sqrt{5}) \rfloor$, while in case (ii) we have from (10) that $n = \lfloor q(3+\sqrt{5}) \rfloor$

This proves that n appears among the x_k, y_k. We now argue that the x_k, y_k are distinct.

In Case (i) we can that since $\beta > 1$ is as small as possible, $n \neq y_k$ for every k. In Case (ii) we see that $n \neq x_k$ for every k.

So if an n appears twice, then we would have (a) $x_k = x_\ell$ or (b) $y_k = y_\ell$ for some $k > \ell$.
But (a) implies $0 = x_k - x_\ell = \frac{1}{2}(k-\ell)(1+\sqrt{5}) - \eta$ where $|\eta| < 1$, a contradiction. We rule out (b) in the same way.

Geography

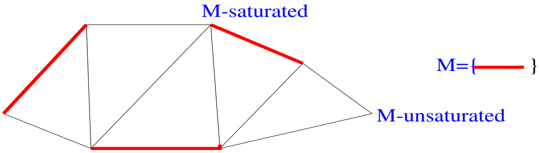
Start with a chip sitting on a vertex v of a graph or digraph G. A move consists of moving the chip to a neighbouring vertex.

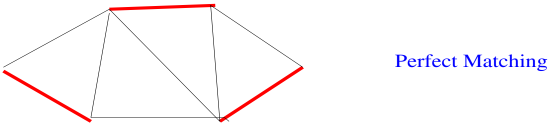
In edge geography, moving the chip from x to y deletes the edge (x, y). In vertex geography, moving the chip from x to y deletes the vertex x.

The problem is given a position (G, v), to determine whether this is a P or N position.

Complexity Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.

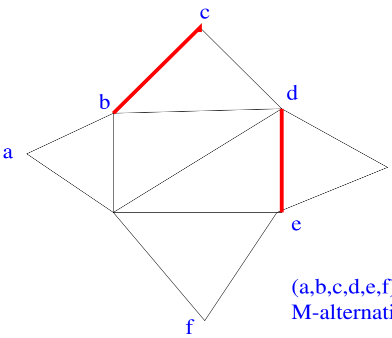
We need some simple results from the theory of matchings on graphs.
A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





Undirected Vertex Geography

M-alternating path



(a,b,c,d,e,f) is an
M-alternating path

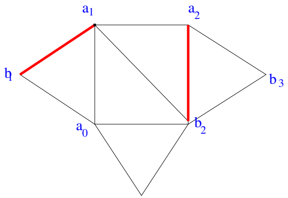
An *M*-alternating path joining 2 *M*-unsaturated vertices is called an *M*-augmenting path.

M is a *maximum* matching of G if no matching M' has more edges.

Theorem

M is a maximum matching iff M admits no M-augmenting paths.

Proof Suppose M has an augmenting path
$P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where
$e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and
$f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



Let $M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}$.

Undirected Vertex Geography

- $|M'| = |M| + 1$.
- M' is a matching

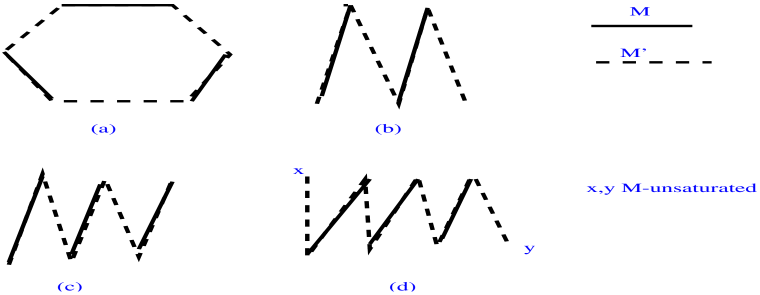
For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

Undirected Vertex Geography

Suppose M is not a maximum matching and $|M'| > |M|$.
Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.
Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



$|M'| > |M|$ implies that there is at least one path of type (d).
Such a path is M-augmenting

Undirected Vertex Geography

Theorem

(G, v) is an N-position in UVG iff every maximum matching of G covers v.

Proof (i) Suppose that M is a maximum matching of G which covers v. Player 1's strategy is now: Move along the M-edge that contains the current vertex.

If Player 1 were to lose, then there would exist a sequence of edges $e_1, f_1, \ldots, e_k, f_k$ such that $v \in e_1$, $e_1, e_2, \ldots, e_k \in M$, $f_1, f_2, \ldots, f_k \notin M$ and $f_k = (x, y)$ where y is the current vertex for Player 1 and y is not covered by M.

But then if $A = \{e_1, e_2, \ldots, e_k\}$ and $B = \{f_1, f_2, \ldots, f_k\}$ then $(M \setminus A) \cup B$ is a maximum matching (same size as M) which does not cover v, contradiction.

(ii) Suppose now that there is some maximum matching M which does not cover v. If (v, w) is Player 1's move,then w

must be covered by M, else M is not a maximum matching.

Player 2's strategy is now: Move along the M-edge that contains the current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y)$ where y is the current vertex for Player 2 and y is not covered by M.

But then we have defined an augmenting path from v to y and so M is not a maximum matching, contradiction. □

Undirected Vertex Geography

Note that we can determine whether or not v is covered by all maximum matchings as follows: Find the size σ of the maximum matching G.

This can be done in $O(n^3)$ time on an n-vertex graph. Find the size σ' of a maximum matching in $G - v$. Then v is covered by all maximum matchings of G iff $\sigma \neq \sigma'$.

Undirected Edge Geography on a bipartite graph

An *even kernel* of G is a non-empty set $S \subseteq V$ such that (i) S is an independent set and (ii) $v \notin S$ implies that $deg_S(v)$ is even, (possibly zero). ($deg_S(v)$ is the number of neighbours of v in S.)

Lemma

If S is an even kernel and $v \in S$ then (G, v) is a P-position in UEG.

Proof Any move at a vertex in S takes the chip outside S and then Player 2 can immediately put the chip back in S. After a move from $x \in S$ to $y \notin S$, $deg_S(y)$ will become odd and so there is an edge back to S. making this move, makes $deg_S(y)$ even again. Eventually, there will be no $S : \bar{S}$ edges and Player 1 will be stuck in S. □

We now discuss Bipartite UEG i.e. we assume that G is bipartite, G has bipartion consisting of a copy of $[m]$ and a disjoint copy of $[n]$ and edges set E. Now consider the $m \times n$ 0-1 matrix A with $A(i,j) = 1$ iff $(i,j) \in E$.

We can play our game on this matrix: We are either positioned at row i or we are positioned at column j. If say, we are positioned at row i, then we choose a j such that $A(i,j) = 1$ and (i) make $A(i,j) = 0$ and (ii) move the position to column j. An analogous move is taken when we positioned at column j.

Lemma

Suppose the current position is row i. This is a P-position iff row i is in the span of the remaining rows (is the sum (mod 2) of a subset of the other rows) or row i is a zero row. A similar statement can be made if the position is column j.

Proof If row i is a zero row then vertex i is isolated and this is clearly a P-position. Otherwise, assume the position is row 1 and there exists $I \subseteq [m]$ such that $1 \in I$ and

$$r_1 = \sum_{i \in I \setminus \{1\}} r_i (mod\ 2) \text{ or } \sum_{i \in I} r_i = 0 (mod\ 2) \tag{9}$$

where r_i denotes row i.

I is an even kernel: If $x \notin I$ then either (i) x corresponds to a row and there are no x, I edges or (ii) x corresponds to a column and then $\sum_{i \in I} A(i, x) = 0 (mod\ 2)$ from (9) and then x has an even number of neighbours in I.

Undirected Edge Geography on a bipartite graph

Now suppose that (9) does not hold for any I. We show that there exists a ℓ such that $A(1, \ell) = 1$ and putting $A(1, \ell) = 0$ makes column ℓ dependent on the remaining columns. Then we will be in a P-position, by the first part.

Let e_1 be the m-vector with a 1 in row 1 and a 0 everywhere else. Let A^* be obtained by adding e_1 to A as an $(n + 1)$th column. Now the row-rank of A^* is the same as the row-rank of A (here we are doing all arithmetic modulo 2). Suppose not, then if r_i^* is the ith row of A^* then there exists a set J such that

$$\sum_{i \in J} r_i = 0 (mod\ 2) \neq \sum_{i \in J} r_i^* (mod\ 2).$$

Now $1 \notin J$ because r_1 is independent of the remaining rows of A, but then $\sum_{i \in J} r_i = 0(mod\ 2)$ implies $\sum_{i \in J} r_i^* = 0(mod\ 2)$ since the last column has all zeros, except in row 1.

Thus rank A^* = rank A and so there exists $K \subseteq [n]$ such that

$$e_1 = \sum_{k \in K} c_k (mod\ 2) \text{ or } e_1 + \sum_{k \in K} c_k = 0 (mod\ 2) \qquad (10)$$

where c_k denotes column k of A.

Thus there exists $\ell \in K$ such that $A(1,\ell) = 1$. Now let $c'_j = c_j$ for $j \neq \ell$ and c'_ℓ be obtained from c_ℓ by putting $A(1,\ell) = 0$ i.e. $c'_\ell = c_\ell + e_1$. But then (10) implies that $\sum_{k \in K} c'_k = 0(mod\ 2)$ ($K = \{k\}$ is a possibility here).. □

Tic Tac Toe

We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English).

The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1, x_2, \ldots, x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(d)})$, $j = 1, 2, \ldots, n$ where each sequence $x^{(i)}$ is either (i) of the form $k, k, \ldots, k$ for some $k \in [n]$ or is (ii) $1, 2, \ldots, n$ or is (iii) $n, n-1, \ldots, 1$. Finally, we cannot have Case (i) for all i.

Thus in the (familiar) 3×3 case, the top row is defined by $x^{(1)} = 1, 1, 1$ and $x^{(2)} = 1, 2, 3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)} = 3, 2, 1$ and $x^{(2)} = 1, 2, 3$

Lemma

The number of winning lines in the (n, d) game is $\frac{(n+2)^d - n^d}{2}$.

Proof In the definition of a line there are n choices for k in (i) and then (ii), (iii) make it up to $n + 2$. There are d independent choices for each i making $(n + 2)^d$.

Now delete n^d choices where only Case (i) is used. Then divide by 2 because replacing (ii) by (iii) and vice-versa whenever Case (i) does not hold produces the same set of points (traversing the line in the other direction). □

Tic Tac Toe

The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on.

A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a wnning strategy:

Lemma

Player 1 can always get at least a draw.

Proof We prove this by considering *strategy stealing*.

Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move x_1. Player 2 will then move with y_1. Player 1 will now win playing the winning strategy for Player 2 against a first move of y_1.

This can be carried out until the strategy calls for move x_1 (if at all). But then Player 1 can make an arbitrary move and continue, since x_1 has already been made. □

The Hales-Jewett Theorem of Ramsey Theory implies that there is a winner in the (n, d) game, when n is large enough with respect to d. The winner is of course Player 1.

Tic Tac Toe

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

The above array gives a strategy for Player 2 in the 5×5 game ($d = 2, n = 5$).

For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number i, then Player 2 responds by choosing the other cell with the number i.

This ensures that Player 1 cannot take line i. If Player 1 chooses the * then Player 2 can choose any cell with an unused number.

Tic Tac Toe

So, later in the game if Player 1 chooses a cell with j and Player 2 already has the other j, then Player 2 can choose an arbitrary cell.

Player 2's strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells asociated with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.

We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of A and giving it his colour.

A player wins if one of the sets A_i is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$.

This is called a *draw forcing pairing*. Player 2 responds to Player 1's choice of $x_{2i+\delta}, \delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from X, then Player 2 can choose any uncoloured element of X.

Tic Tac Toe

In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs x_{2i-1}, x_{2i} and so Player 1 cannot have completely coloured A_i for $i = 1, 2, \ldots, N$.

Theorem

If

$$\left|\bigcup_{X\in\mathcal{G}} X\right| \geq 2|\mathcal{G}| \qquad \forall \mathcal{G}\subseteq\mathcal{F} \tag{11}$$

then there is a draw forcing pairing.

Proof We define a bipartite graph Γ. A will be one side of the bipartition and $B=\{b_1,b_2,\ldots,b_{2N}\}$. Here b_{2i-1} and b_{2i} both represent A_i in the sense that if $a\in A_i$ then there is an edge (a,b_{2i-1}) and an edge (a,b_{2i}).

A draw forcing pairing corresponds to a complete matching of B into A and the condition (11) implies that Hall's condition is satisfied. □

Corollary

If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.

Proof The degree of $a \in A$ is at most $2(n/2)$ in Γ and the degree of each $b \in B$ is at least n. This implies (via Hall's condition) that there is a complete matching of B into A. □

Tic Tac Toe

Consider Tic tac Toe when $d = 2$. If n is even then every array element is in at most 3 lines (one row, one column and at most one diagonal) and if n is odd then every array element is in at most 4 lines (one row, one column and at most two diagonals).

Thus there is a draw forcing pairing if $n \geq 6$, n even and if $n \geq 9$, n odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.)

Tic Tac Toe

In general we have

Lemma

If $n \geq 3^d - 1$ and n is odd or if $n \geq 2^d - 1$ and n is even, then there is a draw forcing pairing of (n, d) Tic tac Toe.

Proof We only have to estimate the number of lines through a fixed point $\mathbf{c} = (c_1, c_2, \ldots, c_d)$.

If n is odd then to choose a line L through $\mathbf{c}$ we specify, for each index i whether L is (i) constant on i, (ii) increasing on i or (iii) decreasing on i.

This gives 3^d choices. Subtract 1 to avoid the all constant case and divide by 2 because each line gets counted twice this way.

When n is even, we observe that once we have chosen in which positions L is constant, L is determined.

Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is x or $n - x + 1$. Assuming w.l.o.g. that $x \leq n/2$ we see that $x < n - x + 1$ and the positions with x increase together at the same time as the positions with $n - x + 1$ decrease together.

Thus the number of lines through $\mathbf{c}$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. □

Quasi-probabilistic method

We now prove a theorem of Erdős and Selfridge.

Theorem

If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$, then Player 2 can get a draw in the game defined by $\mathcal{F}$.

Proof At any point in the game, let C_j denote the set of elements in A which have been coloured with Player j's colour, $j = 1, 2$ and $U = A \setminus C_1 \cup C_2$. Let

$$\Phi = \sum_{i:A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.$$

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots,$. Then we observe that immediately after Player 1's first move, $\Phi < N2^{-(n-1)} < 1$.

Quasi-probabilistic method

We will show that Player 2 can keep $\Phi < 1$ through out. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i:A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let Φ_j be the value of Φ after the choice of $x_1, y_1, \ldots, x_j$. then if U, C_1, C_2 are defined at precisely this time,

$$\begin{aligned}
\Phi_{j+1} - \Phi_j &= -\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \notin A_i, x_{j+1} \in A_i}} 2^{-|A_i \cap U|} \\
&\leq -\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:A_i \cap C_2 = \emptyset \\ x_{j+1} \in A_i}} 2^{-|A_i \cap U|}
\end{aligned}$$

Quasi-probabilistic method

We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses y_j to maximise $\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y \in A_i}} 2^{-|A_i \cap U|}$ over y.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw. □

In the case of (n, d) Tic Tac Toe, we see that Player 2 can force a draw if

$$\frac{(n+2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for n large, by

$$n \geq (1+\epsilon) d \log_2 d$$

where $\epsilon > 0$ is a small positive constant.