BASIC COUNTING

Let $\phi(m, n)$ be the number of mappings from $[n]$ to $[m]$.

Theorem

$$\phi(m, n) = m^n$$

Proof By induction on n.

$$\phi(m, 0) = 1 = m^0.$$

$$\begin{aligned}\phi(m, n+1) &= m\phi(m, n) \\ &= m \times m^n \\ &= m^{n+1}.\end{aligned}$$

□

$\phi(m, n)$ is also the number of sequences $x_1 x_2 \cdots x_n$ where $x_i \in [m]$, $i = 1, 2, \ldots, n$.

Let $\psi(n)$ be the number of subsets of $[n]$.

Theorem

$$\psi(n) = 2^n.$$

Proof (1) By induction on n.

$\psi(0) = 1 = 2^0$.

$$\begin{aligned}
&\psi(n+1) \\
&= \#\{\text{sets containing } n+1\} + \#\{\text{sets not containing } n+1\} \\
&= \psi(n) + \psi(n) \\
&= 2^n + 2^n \\
&= 2^{n+1}.
\end{aligned}$$

There is a general principle that if there is a 1-1 correspondence between two finite sets A, B then $|A| = |B|$. Here is a use of this principle.

Proof (2).
For $A \subseteq [n]$ define the map $f_A : [n] \to \{0, 1\}$ by

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

f_A is the characteristic function of A.

Distinct A's give rise to distinct f_A's and vice-versa.

Thus $\psi(n)$ is the number of choices for f_A, which is 2^n by Theorem 1. □

Let $\psi_{odd}(n)$ be the number of odd subsets of $[n]$ and let $\psi_{even}(n)$ be the number of even subsets.

Theorem

$$\psi_{odd}(n) = \psi_{even}(n) = 2^{n-1}.$$

Proof For $A \subseteq [n-1]$ define

$$A' = \begin{cases} A & |A| \text{ is odd} \\ A \cup \{n\} & |A| \text{ is even} \end{cases}$$

The map $A \to A'$ defines a bijection between $[n-1]$ and the odd subsets of $[n]$. So $2^{n-1} = \psi(n-1) = \psi_{odd}(n)$. Futhermore,

$$\psi_{even}(n) = \psi(n) - \psi_{odd}(n) = 2^n - 2^{n-1} = 2^{n-1}.$$

Let $\phi_{1-1}(m,n)$ be the number of 1-1 mappings from $[n]$ to $[m]$.

Theorem

$$\phi_{1-1}(m,n) = \prod_{i=0}^{n-1}(m-i). \quad (1)$$

Proof Denote the RHS of (1) by $\pi(m,n)$. If $m < n$ then $\phi_{1-1}(m,n) = \pi(m,n) = 0$. So assume that $m \geq n$. Now we use induction on n.

If $n = 0$ then we have $\phi_{1-1}(m,0) = \pi(m,0) = 1$.

In general, if $n < m$ then

$$\begin{aligned} \phi_{1-1}(m,n+1) &= (m-n)\phi_{1-1}(m,n) \\ &= (m-n)\pi(m,n) \\ &= \pi(m,n+1). \end{aligned}$$

□

$\phi_{1-1}(m,n)$ also counts the number of length n ordered sequences distinct elements taken from a set of size m.

$$\phi_{1-1}(n,n) = n(n-1)\cdots 1 = n!$$

is the number of ordered sequences of $[n]$ i.e. the number of permutations of $[n]$.

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$$

Let X be a finite set and let

$$\binom{X}{k} \text{ denote the collection of } k\text{-subsets of } X.$$

Theorem

$$\left|\binom{X}{k}\right| = \binom{|X|}{k}.$$

Proof Let $n = |X|$,

$$k!\left|\binom{X}{k}\right| = \phi_{1-1}(n,k) = n(n-1)\cdots(n-k+1).$$

Let m, n be non-negative integers. Let Z_+ denote the non-negative integers. Let

$$S(m,n) = \{(i_1, i_2, \ldots, i_n) \in Z_+^n : \ i_1 + i_2 + \cdots + i_n = m\}.$$

Theorem

$$|S(m,n)| = \binom{m+n-1}{n-1}.$$

Proof imagine $m+n-1$ points in a line. Choose positions $p_1 < p_2 < \cdots < p_{n-1}$ and color these points red. Let $p_0 = 0,\ p_n = m+1$. The gap sizes between the red points

$$i_t = p_t - p_{t-1} - 1,\ t = 1, 2, \ldots, n$$

form a sequence in $S(m,n)$ and vice-versa. □

$|S(m,n)|$ is also the number of ways of coloring m *indistinguishable* balls using n colors.

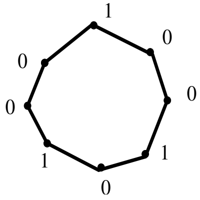
Suppose that we want to count the number of ways of coloring these balls so that each color appears at least once i.e. to compute $|S(m,n)^*|$ where, if $N = \{1, 2, \ldots, \}$

$$
\begin{aligned}
& S(m,n)^* = \\
& \{(i_1, i_2, \ldots, i_n) \in N^n : \; i_1 + i_2 + \cdots + i_n = m\} \\
& = \{(i_1 - 1, i_2 - 1, \ldots, i_n - 1) \in Z_+^n : \\
& \qquad\qquad (i_1 - 1) + (i_2 - 1) + \cdots + (i_n - 1) = m - n\}
\end{aligned}
$$

Thus,

$$|S(m,n)^*| = \binom{m-n+n-1}{n-1} = \binom{m-1}{n-1}.$$

Seperated 1's on a cycle



How many ways (patterns) are there of placing k 1's and $n-k$ 0's at the vertices of a polygon with n vertices so that no two 1's are adjacent?

Choose a vertex v of the polygon in n ways and then place a 1 there. For the remainder we must choose $a_1, \ldots, a_k \geq 1$ such that $a_1 + \cdots + a_k = n - k$ and then go round the cycle (clockwise) putting a_1 0's followed by a 1 and then a_2 0's followed by a 1 etc..

Each pattern π arises k times in this way. There are k choices of v that correspond to a 1 of the pattern. Having chosen v there is a unique choice of $a_1, a_2, \ldots, a_k$ that will now give π.

There are $\binom{n-k-1}{k-1}$ ways of choosing the a_i and so the answer to our question is

$$\frac{n}{k}\binom{n-k-1}{k-1.}$$

Theorem

Symmetry

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof Choosing r elements to include is equivalent to choosing $n-r$ elements to exclude. □

Theorem

Pascal's Triangle

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof A $k+1$-subset of $[n+1]$ either
(i) includes $n+1$ —— $\binom{n}{k}$ choices or
(ii) does not include $n+1$ —— $\binom{n}{k+1}$ choices.

Pascal's Triangle

The following array of binomial coefficents, constitutes the famous triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

. . .

Theorem

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \qquad (2)$$

Proof 1: Induction on n for arbitrary k.
Base case: $n = k$; $\binom{k}{k} = \binom{k+1}{k+1}$
Inductive Step: assume true for $n \geq k$.

$$\begin{aligned} \sum_{m=k}^{n+1} \binom{m}{k} &= \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k} \\ &= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{Induction} \\ &= \binom{n+2}{k+1}. \quad \text{Pascal's triangle} \end{aligned}$$

Proof 2: Combinatorial argument.

If S denotes the set of $k+1$-subsets of $[n+1]$ and S_m is the set of $k+1$-subsets of $[n+1]$ which have largest element $m+1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of S.
- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.

□

Theorem

Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}.$$

Proof Split $[m+n]$ into $A = [m]$ and $B = [m+n] \setminus [m]$. Let S denote the set of k-subsets of $[m+n]$ and let $S_r = \{X \in S : |X \cap A| = r\}$. Then

- $S_0, S_1, \ldots, S_k$ is a partition of S.
- $|S_0| + |S_1| + \cdots + |S_k| = |S|$.
- $|S_r| = \binom{m}{r}\binom{n}{k-r}$.
- $|S| = \binom{m+n}{k}$.

□

Theorem

Binomial Theorem

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Proof Coefficient x^r in $(1+x)(1+x)\cdots(1+x)$: choose x from r brackets and 1 from the rest. □

Applications of Binomial Theorem

- $x = 1$:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

LHS counts the number of subsets of all sizes in $[n]$.

- $x = -1$:

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1-1)^n = 0,$$

i.e.

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

and number of subsets of even cardinality = number of subsets of odd cardinality.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Differentiate both sides of the Binomial Theorem w.r.t. x.

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}.$$

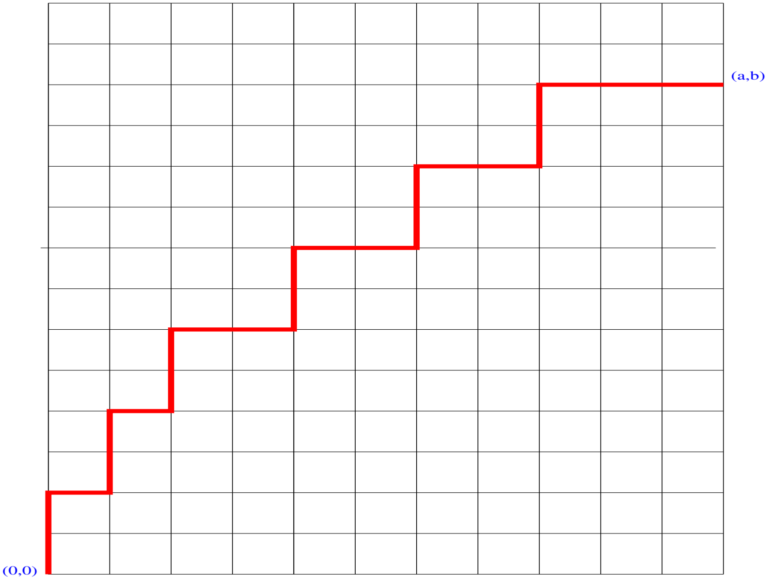
Now put $x = 1$.

Grid path problems

A *monotone path* is made up of segments $(x, y) \to (x + 1, y)$ or $(x, y) \to (x, y + 1)$.

$(a, b) \to (c, d)$)= {monotone paths from (a, b) to (c, d)}.

We drop the $(a, b) \to$ for paths starting at $(0, 0)$.


(a,b)
(0,0)

We consider 3 questions: Assume $a, b \geq 0$.

1. How large is $PATHS(a, b)$?

2. Assume $a < b$. Let $PATHS_{>}(a, b)$ be the set of paths in $PATHS(a, b)$ which do not touch the line $x = y$ except at $(0, 0)$. How large is $PATHS_{>}(a, b)$?

3. Assume $a \leq b$. Let $PATHS_{\geq}(a, b)$ be the set of paths in $PATHS(a, b)$ which do not pass through points with $x > y$. How large is $PATHS_{\geq}(a, b)$?

1. $STRINGS(a, b) = \{x \in \{\mathrm{R}, \mathrm{U}\}^* : \ x$ has a R's and b U's$\}$. [1]

There is a natural bijection between $PATHS(a, b)$ and $STRINGS(a, b)$:

Path moves to Right, add R to sequence.
Path goes up, add U to sequence.

So

$$|\mathrm{PATHS}(a, b)| = |\mathrm{STRINGS}(a, b)| = \binom{a+b}{a}$$

since to define a string we have state which of the $a + b$ places contains an R.

[1] $\{\mathrm{R}, \mathrm{U}\}^*$ = set of strings of R's and U's

2. Every path in $\mathrm{PATHS}_{>}(a,b)$ goes through (0,1). So

$$|PATHS_{>}(a,b)| = |PATHS((0,1) \to (a,b))| - |\mathrm{PATHS}_{\not>}((0,1) \to (a,b))|.$$

Now

$$|PATHS((0,1) \to (a,b))| = \binom{a+b-1}{a}$$

and

$$|PATHS_{\not>}((0,1) \to (a,b))| = |PATHS((1,0) \to (a,b))| = \binom{a+b-1}{a-1}.$$

We explain the first equality momentarily. Thus

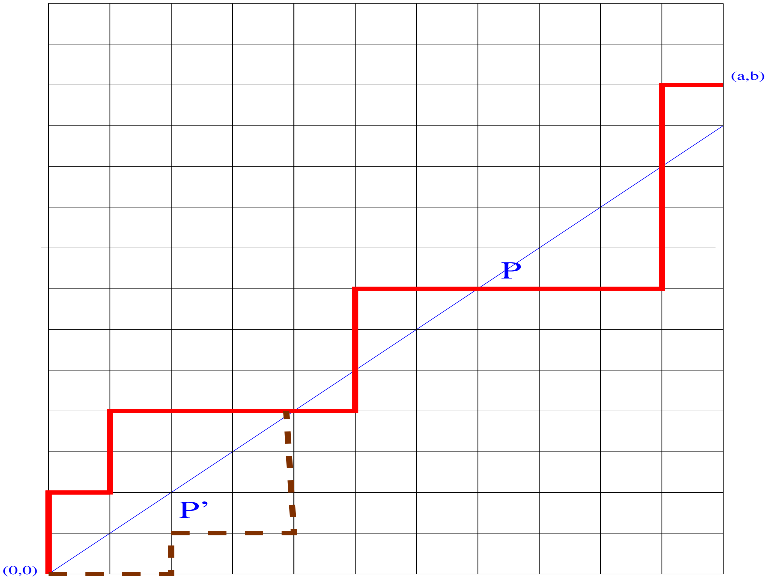
$$\begin{aligned} |\mathrm{PATHS}_{>}(a,b)| &= \binom{a+b-1}{a} - \binom{a+b-1}{a-1} \\ &= \frac{b-a}{a+b}\binom{a+b}{a}. \end{aligned}$$

Suppose $P \in \mathrm{PATHS}_{\not\geq}((0,1) \to (a,b))$. We define $P' \in \mathrm{PATHS}((1,0) \to (a,b))$ in such a way that $P \to P'$ is a bijection.

Let (c,c) be the first point of P, which lies on the line $L = \{x = y\}$ and let S denote the initial segment of P going from $(0,1)$ to (c,c).

P' is obtained from P by deleting S and replacing it by its reflection S' in L.

To show that this defines a bijection, observe that if $P' \in \mathrm{PATHS}((1,0) \to (a,b))$ then a similarly defined *reverse reflection* yields a $P \in \mathrm{PATHS}_{\not\geq}((0,1) \to (a,b))$.


(a,b)
P
P'
(0,0)

3. Suppose $P \in \mathrm{PATHS}_{\geq}(a, b)$. We define $P'' \in \mathrm{PATHS}_{>}(a, b+1)$ in such a way that $P \to P''$ is a bijection.

Thus

$$|PATHS_{\geq}(a, b)| = \frac{b-a+1}{a+b+1}\binom{a+b+1}{a}.$$

In particular

$$\begin{aligned} |\mathrm{PATHS}_{\geq}(a, a)| &= \frac{1}{2a+1}\binom{2a+1}{a} \\ &= \frac{1}{a+1}\binom{2a}{a}. \end{aligned}$$

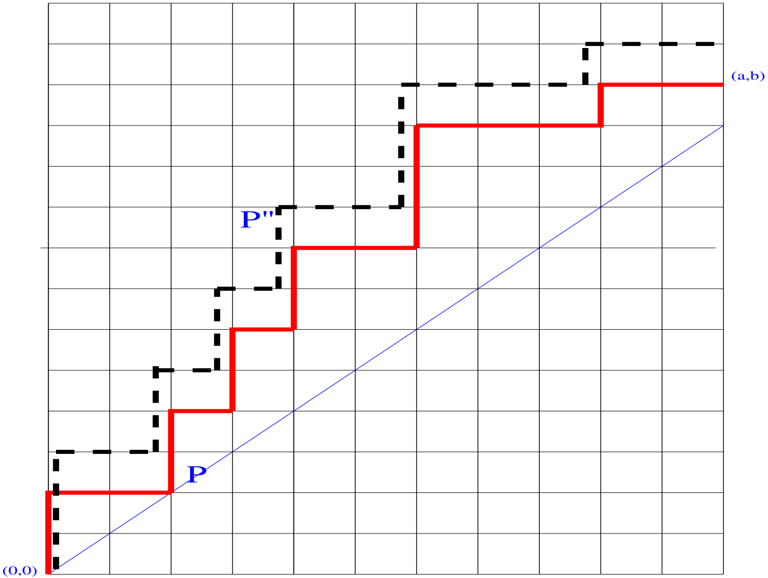
The final expression is called a *Catalan Number*.

The bijection

Given P we obtain P'' by *raising it vertically one position and then adding the segment* $(0,0) \to (0,1)$.

More precisely, if $P = (0,0),(x_1,y_1),(x_2,y_2),\ldots,(a,b)$ then $P'' = (0,0),(0,1),(x_1,y_1+1),\ldots,(a,b+1)$.

This is clearly a $1-1$ onto function between $\mathrm{PATHS}_{\geq}(a,b)$ and $\mathrm{PATHS}_{>}(a,b+1)$.


(a,b)
P''
P
(0,0)

Multi-sets

Suppose we allow elements to appear several times in a set: $\{a, a, a, b, b, c, c, c, d, d\}$.

To avoid confusion with the standard definition of a set we write $\{3 \times a, 2 \times b, 3 \times c, 2 \times d\}$.

How many distinct permutations are there of the multiset $\{a_1 \times 1, a_2 \times 2, \ldots, a_n \times n\}$?

Ex. $\{2 \times a, 3 \times b\}$.

aabbb; *ababb*; *abbab*; *abbba*; *baabb*
babab; *babba*; *bbaab*; *bbaba*; *bbbaa*.

Start with $\{a_1, a_2, b_1, b_2, b_3\}$ which has $5! = 120$ permutations:
$\ldots a_2 b_3 a_1 b_2 b_1 \ldots a_1 b_2 a_2 b_1 b_3 \ldots$
After erasing the subscripts each possible sequence e.g. $ababb$ occurs $2! \times 3!$ times and so the number of permutations is $5!/2!3! = 10$.
In general if $m = a_1 + a_2 + \cdots + a_n$ then the number of permutations is

$$\frac{m!}{a_1!a_2!\cdots a_n!}$$

Multinomial Coefficients

$$\binom{m}{a_1, a_2, \ldots, a_n} = \frac{m!}{a_1! a_2! \cdots a_n!}$$

$$(x_1 + x_2 + \cdots + x_n)^m =$$

$$\sum_{\substack{a_1+a_2+\cdots+a_n=m \\ a_1 \geq 0, \ldots, a_n \geq 0}} \binom{m}{a_1, a_2, \ldots, a_n} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}.$$

E.g.

$$\begin{aligned}(x_1 + x_2 + x_3)^4 &= \binom{4}{4,0,0} x_1^4 + \binom{4}{3,1,0} x_1^3 x_2 + \\ &\quad \binom{4}{3,0,1} x_1^3 x_3 + \binom{4}{2,1,1} x_1^2 x_2 x_3 + \cdots \\ &= x_1^4 + 4x_1^3 x_2 + 4x_1^3 x_3 + 12x_1^2 x_2 x_3 + \cdots\end{aligned}$$

Contribution of 1 to the coefficient of $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ from every permutation in $S = \{x_1 \times a_1, x_2 \times a_2, \dots, x_n \times a_n\}$.
E.g.

$$(x_1 + x_2 + x_3)^6 = \dots + x_2x_3x_2x_1x_1x_3 + \dots$$

where the displayed term comes by choosing x_2 from first bracket, x_3 from second bracket etc.

Given a permutation $i_1i_2\cdots i_m$ of S e.g. $331422\cdots$ we choose x_3 from the first 2 brackets, x_1 from the 3rd bracket etc. Conversely, given a choice from each bracket which contributes to the coefficient of $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ we get a permutation of S.

Balls in boxes

m distinguishable balls are placed in n distinguishable boxes. Box i gets b_i balls.

$$\# \text{ ways is } \binom{m}{b_1, b_2, \ldots, b_n}.$$

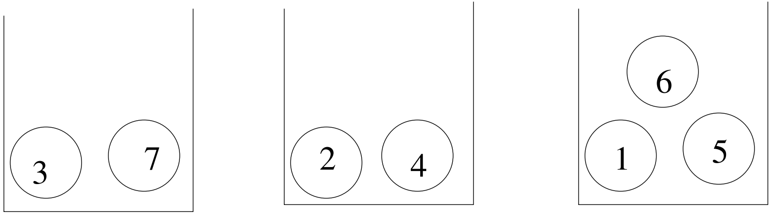
$m = 7, n = 3, b_1 = 2, b_2 = 2, b_3 = 3$
No. of ways is

$$7!/(2!2!3!) = 210$$

$$[1,2][3,4][5,6,7] \quad [1,2][3,5][4,6,7] \quad \cdots \quad [6,7][4,5][1,2,3]$$

3 1 3 2 1 3 2
Ball 1 goes in box 3, Ball 2 goes in box 1, etc.

Conversely, given an allocation of balls to boxes:



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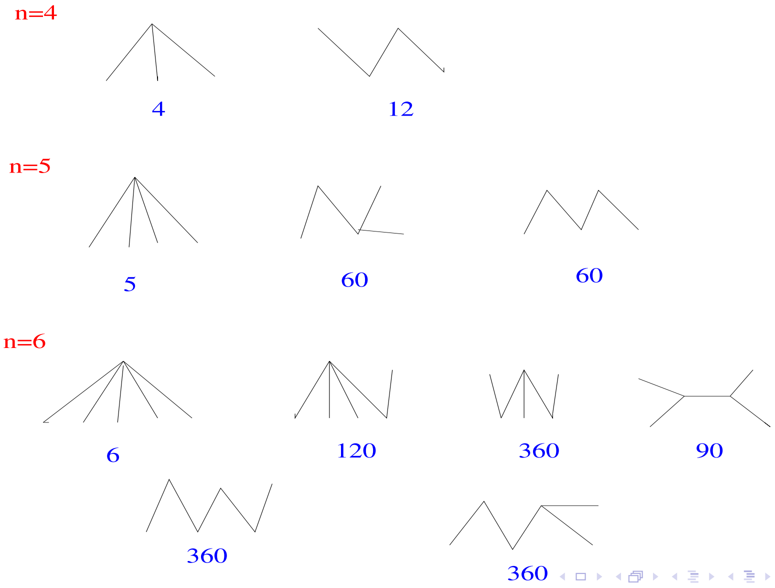
How many trees? – Cayley's Formula



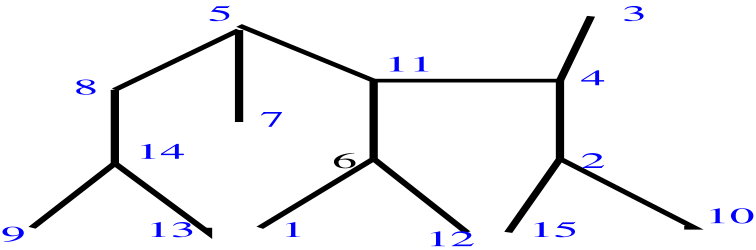
Prüfer's Correspondence

There is a 1-1 correspondence ϕ_V between spanning trees of K_V (the complete graph with vertex set V) and sequences V^{n-2}. Thus for $n \geq 2$

$$\tau(K_n) = n^{n-2} \qquad \text{Cayley's Formula.}$$

Assume some arbitrary ordering $V = \{v_1 < v_2 < \cdots < v_n\}$.

$\phi_V(T)$:
begin
  $T_1 := T$;
  for $i = 1$ **to** $n - 2$ **do**
  begin
    $s_i :=$ neighbour of least leaf ℓ_i of T_i.
    $T_{i+1} = T_i - \ell_i$.
  end $\phi_V(T) = s_1 s_2 \ldots s_{n-2}$
end



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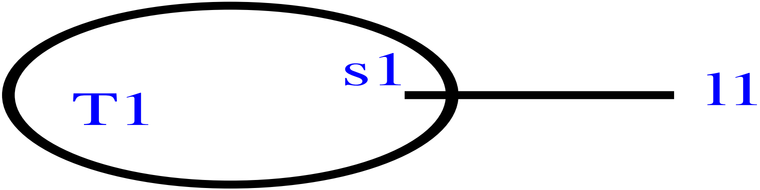
Lemma

$v \in V(T)$ appears exactly $d_T(v) - 1$ times in $\phi_V(T)$.

Proof Assume $n = |V(T)| \geq 2$. By induction on n.
$n = 2$: $\phi_V(T) = \Lambda$ = empty string.
Assume $n \geq 3$:



$\phi_V(T) = s_1\phi_{V_1}(T_1)$ where $V_1 = V - \{s_1\}$.
s_1 appears $d_{T_1}(s_1) - 1 + 1 = d_T(s_1) - 1$ times – induction.
$v \neq s_1$ appears $d_{T_1}(v) - 1 = d_T(v) - 1$ times – induction. □

Construction of ϕ_V^{-1}

Inductively assume that for all $|X| < n$ there is an inverse function ϕ_X^{-1}. (True for $n = 2$).
Now define ϕ_V^{-1} by

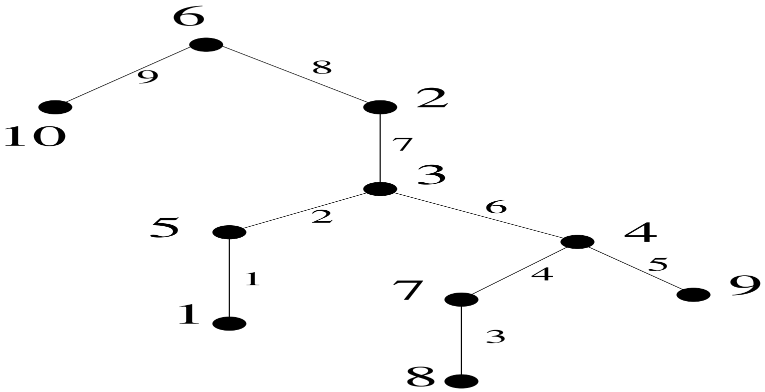
$$\phi_V^{-1}(s_1 s_2 \ldots s_{n-2}) = \phi_{V_1}^{-1}(s_2 \ldots s_{n-2}) \text{ plus edge } s_1\ell_1,$$

where $\ell_1 = \min\{s \in V : \ s \notin \{s_1, s_2, \ldots s_{n-2}\}\}$ and $V_1 = V - \{\ell_1\}$. Then

$$\begin{aligned} \phi_V(\phi_V^{-1}(s_1 s_2 \ldots s_{n-2})) &= s_1 \phi_{V_1}(\phi_{V_1}^{-1}(s_2 \ldots s_{n-2})) \\ &= s_1 s_2 \ldots s_{n-2}. \end{aligned}$$

Thus ϕ_V has an inverse and the correspondence is established.

$n = 10$
$s = 5, 3, 7, 4, 4, 3, 2, 6$.



Number of trees with a given degree sequence

Corollary

If $d_1 + d_2 + \cdots + d_n = 2n - 2$ then the number of spanning trees of K_n with degree sequence $d_1, d_2, \ldots, d_n$ is

$$\binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \frac{(n-2)!}{(d_1 - 1)!(d_2 - 1)! \cdots (d_n - 1)!}.$$

Proof From Prüfer's correspondence this is the number of sequences of length $n - 2$ in which 1 appears $d_1 - 1$ times, 2 appears $d_2 - 1$ times and so on. □

Inclusion-Exclusion

2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

So if $A_1, A_2 \subseteq A$ and $\overline{A}_i = A \setminus A_i$, $i = 1, 2$ then

$$|\overline{A}_1 \cap \overline{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3 sets:

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= |A| - |A_1| - |A_2| - |A_3| \\ &\quad + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

General Case

$A_1, A_2, \ldots, A_N \subseteq A$ and each $x \in A$ has a weight w_x. (In our examples $w_x = 1$ for all x and so $w(X) = |X|$.)

For $S \subseteq [N]$, $A_S = \bigcap_{i \in S} A_i$ and $w(S) = \sum_{x \in S} w_x$.

E.g. $A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18}$.

$A_\emptyset = A$.

Inclusion-Exclusion Formula:

$$w\left(\bigcap_{i=1}^{N} \overline{A}_i\right) = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S).$$

Simple example. How many integers in [1000] are not divisible by 5,6 or 8 i.e. what is the size of $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ below? Here we take $w_x = 1$ for all x.

$$\begin{array}{rcl} A = A_\emptyset & = & \{1, 2, 3, \ldots, \} \\ A_1 & = & \{5, 10, 15, \ldots, \} \\ A_2 & = & \{6, 12, 18, \ldots, \} \\ A_3 & = & \{8, 16, 24, \ldots, \} \\ A_{\{1,2\}} & = & \{30, 60, 90, \ldots, \} \\ A_{\{1,3\}} & = & \{40, 80, 120, \ldots, \} \\ A_{\{2,3\}} & = & \{24, 48, 72, \ldots, \} \\ A_{\{1,2,3\}} & = & \{120, 240, 360, \ldots, \} \end{array} \qquad \begin{array}{l} |A| = 1000 \\ |A_1| = 200 \\ |A_2| = 166 \\ |A_2| = 125 \\ |A_{\{1,2\}}| = 33 \\ |A_{\{1,3\}}| = 25 \\ |A_{\{2,3\}}| = 41 \\ |A_{\{1,2,3\}}| = 8 \end{array}$$

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 1000 - (200 + 166 + 125) \\ &\qquad + (33 + 25 + 41) - 8 \\ &= 600. \end{aligned}$$

Derangements

A derangement of $[n]$ is a permutation π such that

$$\pi(i) \neq i : \; i = 1, 2, \ldots, n.$$

We must express the set of derangements D_n of $[n]$ as the intersection of the complements of sets.
We let $A_i = \{\text{permutations } \pi : \; \pi(i) = i\}$ and then

$$|D_n| = \left| \bigcap_{i=1}^{n} \overline{A}_i \right|.$$

We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n-1)!$: after fixing $\pi(1) = 1$ there are $(n-1)!$ ways of permuting $2, 3, \ldots, n$.

$|A_{\{1,2\}}| = (n-2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n-2)!$ ways of permuting $3, 4, \ldots, n$.

In general

$$|A_S| = (n - |S|)!$$

$$
\begin{aligned}
|D_n| &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.
\end{aligned}
$$

When n is large,

$$
\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \approx e^{-1}.
$$

Proof of inclusion-exclusion formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^{N} \overline{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} w\left(\bigcap_{i=1}^{N} \overline{A}_i\right) &= \sum_{x\in A} w_x (1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) \\ &= \sum_{x\in A} w_x \sum_{S\subseteq[N]} (-1)^{|S|} \prod_{i\in S} \theta_{x,i} \\ &= \sum_{S\subseteq[N]} (-1)^{|S|} \sum_{x\in A} w_x \prod_{i\in S} \theta_{x,i} \\ &= \sum_{S\subseteq[N]} (-1)^{|S|} w(A_S). \end{aligned}$$

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n, other than 1.
$\phi(12) = 4$.
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n.

$$A_i = \{x \in [n] : \; p_i \text{ divides } x\}, \qquad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A}_i \right|$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i} \qquad S \subseteq [k].$$

$$\begin{aligned} \phi(n) &= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} \\ &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \end{aligned}$$

Surjections

Fix n, m. Let

$$A = \{f : [n] \to [m]\}$$

Thus $|A| = m^n$. Let

$$F(n,m) = \{f \in A : \ f \text{ is onto } [m]\}.$$

How big is $F(n,m)$?
Let

$$A_i = \{f \in F : \ f(x) \neq i, \forall x \in [n]\}.$$

Then

$$F(n,m) = \bigcap_{i=1}^{m} \overline{A}_i.$$

For $S \subseteq [m]$

$$\begin{aligned} A_S &= \{f \in A : \ f(x) \notin S, \forall x \in [n]\}. \\ &= \{f : [n] \to [m] \setminus S\}. \end{aligned}$$

So

$$|A_S| = (m - |S|)^n.$$

Hence

$$\begin{aligned} F(n,m) &= \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n. \end{aligned}$$

Scrambled Allocations

We have n boxes $B_1, B_2, \ldots, B_n$ and $2n$ distinguishable balls $b_1, b_2, \ldots, b_{2n}$.
An allocation of balls to boxes, two balls to a box, is said to be *scrambled* if there does **not** exist i such that box B_i contains balls b_{2i-1}, b_{2i}. Let σ_n be the number of scrambled allocations.

Let A_i be the set of allocations in which box B_i contains b_{2i-1}, b_{2i}. We show that

$$|A_S| = \frac{(2(n-|S|))!}{2^{n-|S|}}.$$

Inclusion-Exclusion then gives

$$\sigma_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2(n-k))!}{2^{n-k}}.$$

First consider $A_\emptyset$:

Each permutation π of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box B_i, for $i = 1, 2, \ldots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly 2^n distinct permutations, giving

$$|A_\emptyset| = \frac{(2n)!}{2^n}.$$

To get the formula for $|A_S|$ observe that the contents of $2|S|$ boxes are fixed and so we are in essence dealing with $n - |S|$ boxes and $2(n - |S|)$ balls.

Probléme des Ménages

In how many ways M_n can n male-female couples be seated around a table, alternating male-female, so that no person is seated next to their partner?

Let A_i be the set of seatings in which couple i sit together.

If $|S| = k$ then

$$|A_S| = 2k!(n-k)!^2 \times d_k.$$

d_k is the number of ways of placing k 1's on a cycle of length $2n$ so that no two 1's are adjacent. (We place a person at each 1 and his/her partner on the succeeding 0).

2 choices for which seats are occupied by the men or women. $k!$ ways of assigning the couples to the positions; $(n-k)!^2$ ways of assigning the rest of the people.

$$d_k = \frac{2n}{k}\binom{2n-k-1}{k-1} = \frac{2n}{2n-k}\binom{2n-k}{k}.$$

(See slides 11 and 12).

$$\begin{aligned} M_n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k} \times 2k!(n-k)!^2 \times \frac{2n}{2n-k}\binom{2n-k}{k} \\ &= 2n!\sum_{k=0}^{n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!. \end{aligned}$$

The weight of elements in exactly k sets:

Observe that

$$\prod_{i\in S}\theta_{x,i}\prod_{i\notin S}(1-\theta_{x,i}) = 1 \text{ iff } x\in A_i, i\in S \text{ and } x\notin A_i, i\notin S.$$

W_k is the total weight of elements in exactly k of the A_i:

$$\begin{aligned}
W_k &= \sum_{x\in A} w_x \sum_{|S|=k}\prod_{i\in S}\theta_{x,i}\prod_{i\notin S}(1-\theta_{x,i})\\
&= \sum_{|S|=k}\sum_{x\in A} w_x \prod_{i\in S}\theta_{x,i}\prod_{i\notin S}(1-\theta_{x,i})\\
&= \sum_{|S|=k}\sum_{T\supseteq S}\sum_{x\in A} w_x(-1)^{|T\setminus S|}\prod_{i\in T}\theta_{x,i}\\
&= \sum_{|S|=k}\sum_{T\supseteq S}(-1)^{|T\setminus S|}w(A_T)\\
&= \sum_{\ell=k}^{N}\sum_{|T|=\ell}(-1)^{\ell-k}\binom{\ell}{k}w(A_T).
\end{aligned}$$

As an example. Let $D_{n,k}$ denote the number of permutations π of $[n]$ for which there are exactly k indices i for which $\pi(i) = i$. Then

$$\begin{aligned} D_{n,k} &= \sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} (n-\ell)! \\ &= \sum_{\ell=k}^{n} \frac{n!}{\ell!(n-\ell)!} (-1)^{\ell-k} \frac{\ell!}{k!(\ell-k)!} (n-\ell)! \\ &= \frac{n!}{k!} \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{(\ell-k)!} \\ &= \frac{n!}{k!} \sum_{r=0}^{n-k} \frac{(-1)^r}{r!} \\ &\approx \frac{n!}{ek!} \end{aligned}$$

when n is large and k is constant.

Bonferroni Inequalities

For $\mathbf{x} \in \{0, 1, *\}^N$ let

$$A_{\mathbf{x}} = A_1^{(x_1)} \cap A_2^{(x_2)} \cap \cdots \cap A_n^{(x_N)}.$$

Here

$$A_i^{(x)} = \begin{cases} A_i & x = 1 \\ \bar{A}_i & x = 0 \\ A & x = * \end{cases}.$$

So,

$$A_{0,1,0,*} = \bar{A}_1 \cap A_2 \cap \bar{A}_3 \cap A = \bar{A}_1 \cap A_2 \cap \bar{A}_3.$$

Suppose that $X \subseteq \{0,1,*\}^N$ and

$$\Delta = \Delta(A_1, A_2, \ldots, A_N) = \sum_{\mathbf{x} \in X} \alpha_{\mathbf{x}} |A_{\mathbf{x}}|.$$

Here $\alpha_{\mathbf{x}} \in \boldsymbol{R}$ for $\alpha_{\mathbf{x}} \in X$.

Theorem (Rényi)

$\Delta \geq 0$ for all $A_1, A_2, \ldots, A_N \subseteq A$ iff $\Delta \geq 0$ whenever $A_i = A$ or $A_i = \emptyset$ for $i = 1, 2, \ldots, N$.

Corollary

$$\left| \bigcap_{i=1}^{N} \bar{A}_i \right| - \sum_{i=0}^{k} \sum_{|S|=i} (-1)^i |A_S| \begin{cases} \leq 0 & k \text{ even} \\ \geq 0 & k \text{ odd} \end{cases}$$

Proof of corollary: Suppose that $A_1 = A_2 = \cdots = A_\ell = A$ and $A_{\ell+1} = \cdots = A_N = \emptyset$. If $\ell = 0$ then $\Delta = 0$ and if $0 < \ell \leq N$ then

$$\Delta = 0 - \sum_{i=0}^{k} (-1)^i \binom{\ell}{i} |A|$$
$$= |A| \begin{cases} 0 & k \geq \ell \\ (-1)^{k+1}\binom{\ell-1}{k} & k < \ell. \end{cases}$$

where the identity

$$\sum_{i=0}^{k} (-1)^i \binom{\ell}{i} = (-1)^k \binom{\ell - 1}{k}$$

can be proved by induction on k for $\ell \geq 1$ fixed.

It follows from the corollary that if D_n denotes the number of derangements of $[n]$ then

$$n! \sum_{i=0}^{2k-1} (-1)^i \frac{1}{i!} \leq D_n \leq n! \sum_{i=0}^{2k} (-1)^i \frac{1}{i!},$$

for all $k \geq 0$.

Proof of Rényi's Theorem: We begin by reducing to the case where $X \subseteq \{0,1\}^N$. I.e. we get rid of *-components.

Consider $\mathbf{x} = (0, 1, *, 1)$. We have

$$A_{\mathbf{x}} = A_{(0,1,0,1)} \cup A_{(0,1,1,1)} \text{ and } A_{(0,1,0,1)} \cap A_{(0,1,1,1)} = \emptyset.$$

So,

$$|A_{(0,1,*,1)}| = |A_{(0,1,0,1)}| + |A_{(0,1,1,1)}|.$$

A similar argument gives

$$|A_{(*,1,*,1)}| = |A_{(0,1,0,1)}| + |A_{(0,1,1,1)}| + |A_{(1,1,0,1)}| + |A_{(1,1,1,1)}|.$$

Repeating this we can write

$$\Delta = \sum_{\mathbf{y} \in Y} \alpha_{\mathbf{y}} |A_{\mathbf{y}}| \text{ where } Y \subseteq \{0,1\}^N.$$

We claim now that $\Delta(A_1, A_2, \ldots, A_N) \geq 0$ for all $A_1, A_2, \ldots, A_N \subseteq A$ iff $\alpha_{\mathbf{y}} \geq 0$ for all $\mathbf{y} \in Y$.

Suppose then that $\exists \mathbf{y} = (y_1, y_2, \ldots, y_N) \in Y$ such that $\alpha_{\mathbf{y}} < 0$. Now let

$$A_i = \begin{cases} A & y_i = 1. \\ \emptyset & y_i = 0. \end{cases}$$

Then in this case

$$\Delta(A_1, A_2, \ldots, A_N) = \alpha_{\mathbf{y}} |A| < 0, \text{ contradiction.}$$

For if $\mathbf{y}' = (y'_1, y'_2, \ldots, y'_N)$ and $y'_i \neq y_i$ for some i then $A^{(y'_i)} = \emptyset$ and so $A_{\mathbf{y}'} = \emptyset$ too.