# 21-301 Combinatorics
# Homework 5
# Due: Monday, October 2

1. Fix $k \geq 1$. We say that a family of sets $A_1, A_2, \ldots, A_m \subseteq [n]$ is $k$-intersection safe if there do not exist $i \neq j$ and $\ell_1, \ell_2, \ldots, \ell_k$ such that $i, j \notin \{\ell_1, \ell_2, \ldots, \ell_k\}$ and $A_i \cap A_j \subseteq \bigcup_{t=1}^{k} A_{\ell_t}$. Show that there exist $k$-intersection safe families of size $c_k^n$ for some $c_k > 1$.

   **Solution:** Suppose that we choose our family at random as for the case of $k = 1$. Let $Z_k$ denote the number of $A_1, A_2, \ldots, A_k, B_1, B_2$ such that $B_1 \cap B_2 \subseteq \bigcup_{i=1}^{k} A_i$. Then,

   $$\mathbf{E}(Z) \leq \binom{m}{k+2}\left(1 - \frac{1}{2^{k+2}}\right)^n \leq m^{k+2} e^{-n/2^{k+2}} = \exp\{(k+2)\log m - n/2^{k+2}\} < 1,$$

   if $m < c_k^n$ where $c_k = e^{1/((k+2)2^{k+2})}$.

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every $v$ and $c \in S(v)$ there are at most $d$ neighbors $u$ of $v$ such that $c$ lies in $S(u)$. Use the local lemma to prove that there is a proper coloring of $G$ assigning to each vertex $v$ a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)

   **Solution**: Assume that each list $S(v)$ is of size exactly $10d$. Randomly color each vertex $v$ with a color $c_v$ from its list $S(v)$. For each edge $e = \{v, w\}$ and color $c \in S(v) \cap S(w)$ we let $\mathcal{E}_{e,c}$ be the event that $c_v = c_w = c$. Thus $P(\mathcal{E}_{e.c}) = 1/(10d)^2$.

   Note that $\mathcal{E}_{\{v,w\},c}$ depends only on the colors assigned to $v$ and $w$, and is thus independent of $\mathcal{E}_{\{v',w'\},c'}$ if $\{v', w'\} \cap \{v, w\} = \emptyset$. Hence $\mathcal{E}_{\{v,w\},c}$ only depends on other edges involving $v$ or $w$. Now there are at most $10d \times d$ events $\mathcal{E}_{\{v,w'\},c'}$ where $c' \in S(v) \cap S(w')$. So the maximum degree in the dependency graph is at most $20d^2$. The result follows from $4 \times 20d^2 \times 1/(10d)^2 < 1$.

3. Show that if $4 \cdot \frac{k^2(n-1)}{k-1} \cdot \frac{1}{2^{1-k}} < 1$ then one can 2-color the integers $1, 2, \ldots, n$ such that there is no mono-colored arithmetic progression of length $k$.

   **Solution:** Color the integers randomly. For an arithmetic progression $S = \{a, a + d, \ldots, a + (k-1)d\}$ of length $k$, let $\mathcal{E}_S$ denote the event that $S$ is mono-colored. Then $\Pr(\mathcal{E}_S) = 2^{-(k-1)}$.

   Now consider the dependency graph of these events. $\mathcal{E}_S, \mathcal{E}_T$ are independent if $S, T$ are disjoint. A fixed progression $S$ intersects at most $\frac{k^2(n-1)}{k-1}$ others: choose $x \in S$ in $k$ ways and $x$'s position in $T$ in $k$ ways then choose $d$ in at most $(n-1)/(k-1)$ ways. Now apply the Local Lemma.