21-301 Combinatorics
Homework 11
Due: Wednesday, December 6

1. How many ways are there to k-color an $n \times n$ chessboard when n is odd. The group G is the usual 8 element group e, a, b, c, p, q, r, s.

   Solution: All we need do is compute the number of cycles in the permutations as applied to the chessboard. Let $n = 2m + 1$. In the table, $r \times s$ is short for r cycles of length s.

Permutation	Number of cycles
e	$n^2 \times 1$
a	$m(m+1) \times 4 + 1 \times 1$
b	$2m(m+1) \times 2 + 1 \times 1$
c	$m(m+1) \times 4 + 1 \times 1$
p	$mn \times 2 + n \times 1$
q	$mn \times 2 + n \times 1$
r	$mn \times 2 + n \times 1$
s	$mn \times 2 + n \times 1$

   Thus the number of colorings is

   $$\frac{1}{8}(k^{n^2} + 2 \times k^{m(m+1)+1} + k^{2m(m+1)+1} + 4 \times k^{mn+n}).$$

2. How many ways are there to arrange 2 M's, 4 A's, 5 T's and 6 H's under the condition that any arrangement and its inverse are to be considered the same.

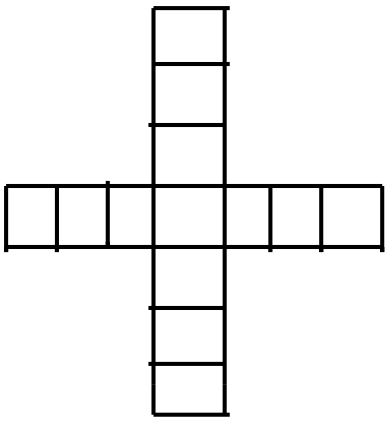
   Solution: The group G consists of $\{e, a\}$ where a is a reflection through the middle of the word. Now

   $$|Fix(e)| = \frac{17!}{2!4!5!6!} = 85765680$$
   $$|Fix(a)| = \frac{8!}{1!2!2!3!} = 1680$$

   A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter T. Then we arrange 1 M, 2 A's, 2 T's and 3 H's in any order and then complete the sequence uniquely to a palindrome.

   Thus by Burnside's theorem, the number of sequences is $\frac{85765680+1680}{2} = 42883680$.

3. How many ways are there of k-coloring the squares of the cross below if the group acting is e_0, e_1, e_2, e_3 where e_j is rotation by $2\pi j/4$. Assume that instead of 13 squares there are $4n + 1$.



Solution:

g	e_0	e_1	e_2	e_3
$\lvert Fix(g)\rvert$	k^{4n+1}	k^{n+1}	k^{2n+1}	k^{n+1}

So the total number of colorings is

$$\frac{k^{4n+1} + k^{n+1} + k^{2n+1} + k^{n+1}}{4}.$$