21-301 Combinatorics
Homework 10
Due: Wednesday, November 29

1. Find the set of P-positions for the take-away games with subtraction sets

   (a) $S = \{1, 3, 7\}$.

   (b) $S = \{1, 4, 6\}$.

   Suppose now that there are two piles and the rules for each pile are as above. Now find the P positions for the two pile game where in one pile S is as in (a) and the other pile is as in (b).

   Solution:

   (a) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10
$g_1(j)$	0	1	0	1	0	1	0	1	0	1	0

   It is apparent that $g_1(j) = j \mod 2$ and this follows by an easy induction: If j is even then $j - x, x \in S$ is odd and if j is odd then $j - x, x \in S$ is even.

   (b) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$g_2(j)$	0	1	0	1	2	0	1	0	1	2	0	1	0	1	2

   So, we see that the pattern 0 1 0 1 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

   (c) The P-positions are those j, k for which $g_1(j) \oplus g_2(k) = 0$.
   Thus

   $$P = \{(j = 0 \mod 2 \text{ and } k = 0, 2 \mod 5) \text{ or } (j = 1 \mod 2 \text{ and } k = 1, 3 \mod 5\}.$$

2. Consider the following game: There is a pile of n chips. A move consists of removing any *proper* factor of n chips from the pile. (For the purposes of this question, a proper factor of n, is any factor, including 1, that is strictly less than n.)The player to leave a pile with one chip wins. Determine the N and P positions and a winning strategy from an N position.

   Solution: n is a P-position iff it is odd. If n is even then the next player can simply remove one chip. If n is odd, then any factor of n is also odd.

3. In a take-away game, the set S of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

   Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.