**Department of Mathematics**
**Carnegie Mellon University**

# 21-301 Combinatorics, Fall 2023: Test 1

Name:____________________________________

Andrew ID:____________________________________

| Problem | Points | Score |
|---|---|---|
| 1 | 33 | |
| 2 | 33 | |
| 3 | 34 | |
| Total | 100 | |

**Q1: (33pts)**
Find a recurrence for the number of sequences in $\{a, b, c\}^n$ that do not contain $aaa$ as a subsequence.
**DO NOT SOLVE THE RECURRENCE.**
**Solution:** Let $B_n$ be the set of sequences in question. Let $B_n^x, x = a, b, c$ be the sequences in $B_n$ that begin with $x$. Then

$$|B_n| = |B_n^{(a)}| + |B_n^{(b)}| + |B_n^{(c)}| = |B_n^{(a)}| + 2|B_{n-1}|.$$

Similarly,

$$|B_n^{(a)}| = |B_n^{(aa)}| + |B_n^{(ab)}| + |B_n^{(ac)}| = |B_n^{(aa)}| + 2|B_{n-2}| = 2|B_{n-3}| + 2|B_{n-2}|.$$

So, the recurrence for $b_n = |B_n|$ is that

$$b_n = 2b_{n-1} + 2b_{n-2} + 2b_{n-3}.$$

**Q2: (33pts)**
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1, a_1 = 9$ and

$$a_n = 6a_{n-1} - 9a_{n-2}$$

for $n \geq 2$.

**(a):** Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

**(b):** Find an expression for $a_n$, $n \geq 0$.

**Solution:** Multiplying the equation by $x^n$ and summing over $n \geq 2$ we obtain

$$(a(x) - 9x - 1) - (6x(a(x) - 1) + (9x^2 a(x)) = 0 \text{ or } a(x) = \frac{3x+1}{(1-3x)^2}.$$

Now

$$\frac{3x}{(1-3x)^2} = \sum_{n=0}^{\infty} n3^n \text{ and } \frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} (n+1)3^n.$$

So

$$a_n = (2n+1)3^n.$$

**Q3: (34pts)**
A table has $4n$ seats. $n$ families sit round the table, so that clockwise we have Man, Woman, Boy, Girl. How many ways are there of seating people so that no family sits completely together?
**Solution:** Let $A_i$ be those sittings where family $i$ sits together. Then if $|S| = s$,

$$|A_S| = \binom{n}{s} s!(n-s)!^4.$$

Explanation: $\binom{n}{s}$ choices of places to seat $S$ and then $s!$ orderings of the families and $(n-s)!^4$ ways of arranging the rest of the people. Thus, the number of possible seatings is

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s}^2 s!(n-s)!^4.$$