Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2021: Test 3

Name:____________________________________

Andrew ID:____________________________________

Write your name and Andrew ID on every page.

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
(a) Consider the following take-away game: There is a pile of n chips. A move consists of removing 2 or 3 chips. Determine the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ and prove that they are what you claim.
(b) Consider the following position in the sum of 3 games described in (a). Determine whether it is an N or P position. There are 3 piles with 19,27,8 chips.
Solution: (a) After looking at the first few numbers $0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 2 \ldots$ one sees that

$$g(n) = \left\lfloor \frac{n \bmod 5}{2} \right\rfloor.$$

We verify this by induction. It is true for $n \leq 10$ by inspection. For $n > 10$ we have that if $n = 5m + s$ then

$$g(n) = mex\{g(n-3), g(n-2)\} = mex\{g(5m+s-3), g(5m+s-2)\}$$

So, by induction

$$g(n) = \begin{cases} mex\{g(5(m-1)+2), g(5(m-1)+3)\} = mex\{1,1\} = 0 & s = 0 \\ mex\{g(5(m-1)+3), g(5(m-1)+4)\} = mex\{1,2\} = 0 & s = 1 \\ mex\{g(5(m-1)+4), g(5m)\} = mex\{2,0\} = 1 & s = 2 \\ mex\{g(5m), g(5m+1)\} = mex\{0,0\} = 1 & s = 3 \\ mex\{g(5m+1), g(5m+2)\} = mex\{0,1\} = 2 & s = 4. \end{cases}$$

The result follows by induction.
(b) $g(19) \oplus g(27) \oplus g(8) = 2 \oplus 1 \oplus 1 = 2$, implying that this is an N position.

Q2: (33pts)
Consider the following game: player A goes first and colors an edge of K_n red. Then player B colors an uncolored edge blue. Then player A colors an uncolored edge red and so on. The game continues until one of the players has completed a monomchromatic triangle. Detemine the smallest n such that one of the players has a winning strategy.

Solution: If $n = 4$ and player A colors {1,2} red then player B can color {3,4} blue. After this, we can assume that player colors {1,3} red. This forces player B to color {2,3} blue and this forces player A to color {3,4} red and then player B finished the game by coloring {1,4} blue. So there is a draw.

If $n = 5$ and player A colors {1,2} red then either (i) player B colors {1,3} blue or (ii) {3,4} blue. In case (i) player A colors {1,4} red and player B is forced to color {2,4} blue. After this, player A colors {1,5} red and has a forced win because B has to close 2 triangles in the next move. In case (ii) player A colors {1,5} red and then B must color {2,5} blue and then A colors {1,3} red and has a forced win.

Q3: (34pts)
Let X denote the set of 2-colorings of $D = [n]$ and let G be the group defined by all permutations of $[n]$. Suppose that G acts on X in the usual way i.e $\pi * c(x) = c(\pi^{-1}(x))$ for any permutation π and coloring $c : D \to \{Red, Blue\}$ and $x \in [n]$.
(a) Determine the structure of the orbits.
(b) How many orbits are there?
(c) Deduce that

$$\sum_{\pi} 2^{\nu(\pi)} = (n+1)!$$

where $\nu(\pi)$ is the number of cycles in the permutation π.
Solution:
(a), (b) The number of elements of each color will be preserved by a permutation. Conversely, if two colorings have the same number of reds and blues then one can obtain one from the other by a permutation. It follows that there are exactly $n+1$ orbits, corresponding to the number of red elements.
(c) For this we apply the Burnside-Frobenius lemma. For a permutation π, $|Fix(\pi)| = 2^{\nu(\pi)}$. So,

$$n + 1 = \frac{1}{n!} \sum_{\pi} |Fix(\pi)| = \frac{1}{n!} \sum_{\pi} 2^{\nu(\pi)}.$$