Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2021: Test 2

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Given a graph $G = (V, E)$ with maximum degree k and a partition of V into sets $V_1, V_2, \ldots, V_m$ of size $10k$, show that there is an independent set in G with one vertex from each set in the partition.
(An independent set S, is a subset of the vertices that contains no edges.)
Solution: let $v_1, v_2, \ldots, v_m$ be chosen randomly from $V_1, V_2, \ldots, V_m$ respectively. For each edge $e = \{x, y\}$ with x, y in distinct parts, let $\mathcal{E}_e$ be the event that x, y are both chosen. Then, $p = \Pr(\mathcal{E}_e) = 1/(10k)^2$. If $x \in V_i$ and $y \in V_j$ then $\mathcal{E}_e$ is independent of $\mathcal{E}_f$, unless $f \cap (V_i \cup V_j) \neq \emptyset$. This means that the dependency graph has maximum degree $d \leq 2k \times 10k$. The result now follows from the symmetric local lemma.

Q2: (33pts)
Let $1 \leq k \leq n/2$ and let $\mathcal{F}$ be a Sperner family consisting of sets of size at most K. Show that $|\mathcal{F}| \leq \binom{n}{k}$.
Solution: let f_i denote the number of sets in $\mathcal{F}$ of size i. Then the LYM inequality implies that

$$\sum_{i=1}^{k} \frac{f_i}{\binom{n}{i}} \leq 1.$$

Now we have $\binom{n}{i} \leq \binom{n}{k}$ for $i \leq k$ and so

$$\sum_{i=1}^{k} \frac{f_i}{\binom{n}{k}} \leq 1,$$

giving us the result.

Q3: (34pts)
We are given 5 integer points $P_i = (x_i, y_i), i = 1, 2, \ldots, 5$ in the plane. Show that there is a pair i, j such that their midpoint $(P_i + P_j)/2$ is also integer. ((a, b) is an integer point if a, b are integers.)
Solution: $(P_i+P_j)/2$ is an integer point if x_i has the same parity as x_j and y_i has the same parity as y_j. Now there are only 4 possible parity pairs for an integer point P, viz. (ODD,ODD), (ODD,EVEN),(EVEN,ODD),(EVEN,EVEN). So, by the pigeon-hole principle there are two points with the same parity pair.