Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2021: Test 1

Name:____________________________________

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Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
How many sequences $a_1, a_2, \ldots, a_m \in [n]$ satisfy $a_i - a_{i-1} \geq k$ for $2 \leq i \leq m$, where k is a positive integer?
(Hint: consider the sequence $x_i = a_i - a_{i-1}, 1 \leq i \leq m+1$, where $a_0 = 0$ and $a_{m+1} = n$.)
Solution: following the hint we see that $x_1 + \cdots + x_{m+1} = x_{m+1} - x_0 = n$ and $x_i \geq k$ for $2 \leq i \leq m$. Then put $y_i = x_i - k, 2 \leq i \leq m, y_1 = x_1 - 1, y_{m+1} = x_{m+1}$ so that $y_1, \ldots, y_{m+1} \geq 0$ and $y_1 + \cdots + y_{m+1} = n - (m-1)k - 1$. There is a 1-1 correspondence between the a_i's and the y_i's and so the answer is $\binom{n-(m-1)k-1+m}{m}$.

Q2: (33pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1, a_1 = 9$ and

$$a_n = 4a_{n-1} - 4a_{n-2}$$

for $n \geq 2$.

(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

(b): Find an expression for a_n, $n \geq 0$.

Solution:

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (4a_{n-1} - 4a_{n-2})\, x^n$$

$$a(x) - 9x - 1 = 4x(a(x) - 1) - 4x^2 a(x)$$

$$a(x) = \frac{5x+1}{4x^2 - 4x + 1} = \frac{5x+1}{(1-2x)^2}$$

$$a(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} (2x)^n + 5x \sum_{n=0}^{\infty} \binom{n+1}{1} (2x)^n$$

$$a_n = (n+1)2^n + 5n2^{n-1} = 2^n + 7n2^{n-1}$$

Q3: (34pts)
n children take off their jackets and shoes and put them into a pile on the floor and go and play. How many ways are there of giving to each of the children a pair of matching shoes and a jacket, so that no child gets his/her own jacket and shoes.
Solution: Suppose that child i is given the jacket of child $\pi_1(i)$ and the shoes of child $\pi_2(i)$. Let

$$A_i = \{(\pi_1, \pi_2) : \ \pi_1(i) = \pi_2(i) = i\}$$

for $i = 1, 2, \ldots, n$.
We need to compute $\left|\bigcap_{i=1}^n \bar{A}_i\right|$. Now if $|S| = k$ then $|A_S| = ((n-k)!)^2$ since we have fixed $\pi_1(i), \pi_2(i)$ for $i \in S$ and the remaining values can be permuted arbitrarily. Thus

$$\left|\bigcap_{i=1}^n \bar{A}_i\right| = \sum_{S \subseteq [N]} (-1)^{|S|}((n-|S|)!)^2 = \sum_{k=0}^n (-1)^k \binom{n}{k} ((n-k)!)^2.$$