Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2020: Test 4

Name:____________________________________

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Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

Q1: (25pts)
Let $G = (V, E)$ be a graph with m edges. Let $\mathcal{C}$ denote the set of cycles of G. For $C \in \mathcal{C}$ we let $|C|$ denote the number of edges in C. Prove that

$$\sum_{C \in \mathcal{C}} \frac{1}{\binom{m}{|C|}} \leq 1.$$

Solution: The set $\mathcal{C}$ is a Sperner family. If $C_1, C_2 \in \mathcal{C}$ then $C_1 \nsubseteq C_2$. The inequality follows directly from the LYM inequality.

Q2: (25pts)
Let $E = \{e_1, \ldots, e_m\}$ and suppose that $S_j \subseteq E, j = 1, \ldots, n$ contains s_j elements. Suppose also that $e_i, i = 1, \ldots, m$ occurs in r_i of the sets S_j. Let $S = \sum_{j=1}^{n} s_j = \sum_{i=1}^{m} r_i$ and $M = \max\{r_1, \ldots, r_m, s_1, \ldots, s_n\}$. Show that if $S > (n-1)M$ then there exist distinct $e_{i_t}, t = 1, \ldots, n$ such that $e_{i_t} \in S_t, t = 1, \ldots, n$.
(Hint: Hall's theorem)
Solution: consider the bipartite graph Γ with vertices $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ where edge (a_i, b_j) exists iff $e_j \in S_i$. We have to prove that Γ contains a matching of A into B. If there is no such matching, then there exists $X \subseteq A$ such that $|N(X)| < |X|$.

$$\ell = \left| \bigcup_{i \in X} S_i \right| < k = |X|.$$

This implies that there is an element $e \in E$ that occurs in at least p of the sets $S_i, i \in X$ where $p\ell \geq \sum_{i \in X} s_j$. By assumption we have

$$\sum_{i \in X} s_i + \sum_{i \notin X} s_i > (n-1)M.$$

Thuis

$$p\ell > (n-1)M - \sum_{i \notin X} s_i \geq (n-1)M - (n-k)M = (k-1)M.$$

Because $\ell \leq k-1$, this implies that $p > M$, a contradiction.

Q3: (25pts)
Find the set of P-positions for the take-away games with subtraction sets

(a) $S = \{1, 3, 7\}$.

(b) $S = \{1, 4, 6\}$.

(Reminder: in a take-away game with subtraction set S, a player can only remove x from a pile, if $x \in S$.)
Suppose now that there are two piles and the rules for each pile are as above. Now find the P positions for the two pile game where in one pile S is as in (a) and the other pile is as in (b)
Solution: let g_a, g_b denote the SG-numbers for the two games. We have

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g_a(n)$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$g_b(n)$	0	1	0	1	2	0	1	0	1	2	0	1	0	1	2	0

An easy induction shows that

$$g_a(n) = n \mod 2 \text{ and } g_b(n) = \begin{cases} 0 & n = 0, 2 \mod 5. \\ 1 & n = 1, 3 \mod 5. \\ 2 & n = 4 \mod 5. \end{cases}$$

The P-posiitons for the two-pile game are when $g_a(bn \oplus g_b(n) = 0$ or

$$P = \{n : n \mod 10 \in \{0, 1, 2, 3, 4\}\}.$$

Q4: (25pts)
How many ways are there to arrange 4C's, 4 G's, 5 A's and 8T's under the condition that any arrangement and its reverse/inverse are to be considered the same.
Solution: The group G consists of $\{e, a\}$ where a is a reflection through the middle of the word. Now

$$\begin{aligned} |Fix(e)| &= \frac{21!}{4!4!5!8!}. \\ |Fix(a)| &= \frac{10!}{2!2!2!4!}. \end{aligned}$$

A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter A. Then we arrange 2 C's, 2 G's, 2 A''s and 4 T's in any order and then complete the sequence uniquely to a palindrome.
The total number of arrangements is $(|Fix(e)| + |Fix(a)|)/2$.