Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2020: Test 3

Name:____________________________________

Andrew ID:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
A set A of $\{0,1\}$ strings of length n is said to be (n,k)**-universal** if for any subset of k coordinates $S = \{i_1, i_2, \ldots, i_k\}$ the projection

$$A_S = \{(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) : \ (a_1, a_2, \ldots, a_n) \in A\}$$

contains all possible 2^k strings of length k. Show that if $\binom{n}{k} 2^k (1 - 2^{-k})^r < 1$ then there is an (n,k)-universal set of size r.
Solution: Let A be a set of r random strings. If A is not (n,k)-universal then there exists a set S of size k and an $x \in \{0,1\}^k$ that is not the projection of a string in A. Let $\mathcal{E}_{S,x}$ denote this event. The union bound implies that the probability of this is at most

$$\sum_S \sum_x \Pr(\mathcal{E}_{S,x}) = \binom{n}{k} 2^k \left(1 - \frac{1}{2^k}\right)^r < 1.$$

This implies the existence of an (n,k)-universal set of size r. by assumption.

Q2: (33pts)
Let $G = (V, E)$ be a graph of maximum degree d. Let $V_1, V_2, \ldots, V_r$ be a partition of V such that $|V_i| \geq 10d$ for $i = 1, 2, \ldots, r$. Use the local lemma to show that G contains a set S such that (i) $|S \cap V_i| = 1$ for $i = 1, 2, \ldots, r$ and (ii) S is independent, i.e. contains no edges of G.
Solution: We can remove vertices from each V_i if needed and so we can assume w.l.o.g. that $|V_i| = 10d$ for $i = 1, 2, \ldots, r$. Choose v_i randomly from V_i for $i = 1, 2, \ldots, r$ and let $S = \{v_1, v_2, \ldots, v_r\}$. For an edge $e = \{x, y\} \in E$ we let $\mathcal{E}_e$ be the event that both $x, y \in S$. Thus $\Pr(\mathcal{E}_e) \leq p = \frac{1}{100d^2}$. An event $\mathcal{E}_e$ depends only on events $\mathcal{E}_f$ for which e and f shae a common vertex. Thus the dependency graph has degree at most $20d^2$. So, $4dp \leq \frac{80d^2}{100d^2} < 1$.

Q3: (34pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following:
$a_0 = 1$, $a_1 = 4$ and

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

for $n \geq 2$.
Determine the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and hence find a_n.
Solution

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty}(a_n - 4a_{n-1} + 4a_{n-2})x^n \\ &= (a(x) - 1 - 4x) - 4x(a(x) - 1) + 4x^2 a(x) \\ &= a(x)(1 - 4x + 4x^2) - 1. \end{aligned}$$

So

$$\begin{aligned} a(x) &= \frac{1}{1 - 4x + 4x^2} \\ &= \frac{1}{(1-2x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)2^n x^n. \end{aligned}$$