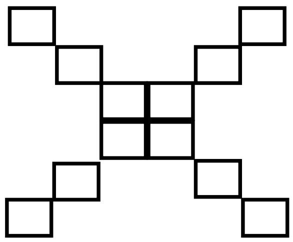
Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2015: Test 4

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)



How many ways are there of k-coloring the squares of the above picture if the group acting is e_0, e_2, p, q where e_j is rotation by $2\pi j/4$ and p, q are horizontal and vertical reflections.
(All small squares are meant to be of the same size here).

Solution

g	e_0	e_2	p	q
$\|Fix(g)\|$	k^{17}	k^9	k^{12}	k^{12}

So the total number of colorings is

$$\frac{k^{17} + k^9 + k^{12} + k^{12}}{4}.$$

Q2: (40pts)
Consider the following take-away game: There is a pile of n chips. A move consists of removing 1 or 4 chips. Determine the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ and prove that they are what you claim.
Solution: After looking at the first few numbers $0, 1, 0, 1, 2, 0, 1, 0, 1, 2, \ldots$ one sees that

$$g(n) = \begin{cases} 0 & n = 0, 2 \mod 5 \\ 1 & n = 1, 3 \mod 5 \\ 2 & n = 4 \mod 5 \end{cases}$$

We verify this by induction. It is true for $n \leq 10$ by inspection. For $n > 10$ we have that if $n = 5m + s$ then

$$g(n) = mex\{g(n-1), g(n-4)\} = mex\{g(5(m-1)+s+4), g(5(m-1)+s+1)\}$$

So, by induction

$$g(n) = \begin{cases} mex\{g(5(m-1)+4), g(5(m-1)+1)\} = mex\{2,1\} = 0 & s = 0 \\ mex\{g(5m), g(5(m-1)+2)\} = mex\{0,0\} = 1 & s = 1 \\ mex\{g(5m+1), g(5(m-1)+3)\} = mex\{1,1\} = 0 & s = 2 \\ mex\{g(5m+2), g(5(m-1)+4)\} = mex\{0,2\} = 1 & s = 3 \\ mex\{g(5m+3), g(5m)\} = mex\{0,1\} = 2 & s = 4 \end{cases}$$

The result follows by induction.

Q3: (20pts)
In the game Split Nim a player removes chips from a non-empty pile and then if desired, has the further option of splitting the reduced pile into two non-empty piles (if the reduced pile has more than one chip). Show that Split Nim has the same N and P positions as ordinary Nim.
Solution: We prove this by induction on the total number t of chips. $t = 0$ is a P position in both games.
Now suppose that $t > 0$ and the position is an N position for Nim. If the player uses regular Nim strategy then the resulting position is a P position for Nim and by induction this is a P position for Split Nim.
Suppose then that $t > 0$ and the position is a P position for Nim. Suppose that the pile sizes are $p_1, p_2, \ldots, p_k$. Suppose that the player first removes chips to leave p_1' chips in the first pile. We know that $p_1' \oplus p_2 \oplus \cdots \oplus p_k \neq 0$ is an N position for Split Nim by induction.
So, suppose that the player now splits the first pile into two piles of size a, b. We will argue that $a \oplus b \oplus p_2 \oplus \cdots \oplus p_k \neq 0$. This is an N position for Nim and it will be an N position for Split Nim by induction. Suppose to the contrary that $a \oplus b \oplus p_2 \oplus \cdots \oplus p_k = 0$. We will argue that $c = a \oplus b \leq a + b$. It follows that the previous position was in fact an N position for Nim, since the player could have removed $p_1 - c$ chips and left a P position.
But if $a = \sum_i a_i 2^i$ and $b = \sum_i b_i 2^i$ then

$$a + b - (a \oplus b) = \sum_i (a_i + b_i - (a_i \oplus b_i))2^i \geq 0.$$