Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2015: Test 3

Name:____________________________________

Andrew ID:____________________________________

Write your name and Andrew ID on every page.

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
The sequence $a_0, a_1, \dots, a_n, \dots$ satisfies the following:
$a_0 = 1$, $a_1 = 4$ and

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

for $n \geq 2$.
Determine the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and hence find a_n.
Solution

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty}(a_n - 4a_{n-1} + 4a_{n-2})x^n \\ &= (a(x) - 1 - 4x) - 4x(a(x) - 1) + 4x^2 a(x) \\ &= a(x)(1 - 4x + 4x^2) - 1. \end{aligned}$$

So

$$\begin{aligned} a(x) &= \frac{1}{1 - 4x + 4x^2} \\ &= \frac{1}{(1 - 2x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)2^n x^n. \end{aligned}$$

Q2: (33pts)
Consider the following take-away game: There is a pile of n chips. A move consists of removing 2 or 3 chips. Determine the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ and prove that they are what you claim.
Solution: After looking at the first few numbers $0, 0, 1, 1, 2, 0, 0, 1, 1, 2, \ldots$ one sees that

$$g(n) = \begin{cases} 0 & n = 0,1 \mod 5 \\ 1 & n = 2,3 \mod 5 \\ 2 & n = 4 \mod 5 \end{cases}$$

We verify this by induction. It is true for $n \leq 10$ by inspection. For $n > 10$ we have that if $n = 5m + s$ then

$$g(n) = mex\{g(n-3), g(n-2)\} = mex\{g(5(m-1)+s+2), g(5(m-1)+s+3)\}$$

So, by induction

$$g(n) = \begin{cases} mex\{g(5(m-1)+2), g(5(m-1)+3)\} = mex\{1,1\} = 0 & s = 0 \\ mex\{g(5(m-1)+3), g(5(m-1)+4)\} = mex\{1,2\} = 0 & s = 1 \\ mex\{g(5(m-1)+4), g(5m)\} = mex\{2,0\} = 1 & s = 2 \\ mex\{g(5m), g(5m+1)\} = mex\{0,0\} = 1 & s = 3 \\ mex\{g(5m+1), g(5m+2)\} = mex\{0,1\} = 2 & s = 4 \end{cases}$$

The result follows by induction.

Q3: (34pts)
Supose that there are m red clubs $R_1, R_2, \dots, R_m$ and m blue clubs $B_1, B_2, \dots, B_m$ in a town of n citizens. Assume that the clubs satisfy the following rules:

- $|R_i \cap B_i|$ is odd for every i;
- $|R_i \cap B_j|$ is even for every $i \neq j$.

Prove that $m \leq n$.
Solution: Let $\mathbf{r}_i, \mathbf{b}_i$ denote the characteristic vectors over the field $\mathbb{F}_2$ of $R_i, B_i i = 1, 2, \dots, m$. We show that these vectors are linearly independent. Suppose for example that

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_m\mathbf{r}_m = 0.$$

Taking the inner product with $\mathbf{b}_j$, we obtain

$$0 = \mathbf{b}_j \cdot \sum_{i=1}^{m} c_i\mathbf{r}_i = \sum_{i=1}^{m} c_i\mathbf{r}_i.\mathbf{b}_j = c_j.$$

It follows that $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ are linearly independent and so $m \leq n$.