Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2015: Test 2

Name:____________________________________

Problem	Points	Score
1	20	
2	40	
3	40	
Total	100	

Q1: (20pts)
Use the Pigeon-Hole Principle to show that if n is odd and π is a permutation of $[n]$ then the product $\prod_{i=1}^{n}(i - \pi(i))$ is even.
Solution: Let $n = 2m + 1$. There are $m + 1$ odd numbers in $[n]$ and m even numbers. By the PHP there must be an odd i such that $\pi(i)$ is also odd. But then $i - \pi(i)$ is even and the product itself is even.

Q2: (40pts)
Let $G = (V, E)$ be a graph of maximum degree d. Let $V_1, V_2, \ldots, V_r$ be a partition of V such that $|V_i| \geq 10d$ for $i = 1, 2, \ldots, r$. Use the local lemma to show that G contains a set S such that (i) $|S \cap V_i| = 1$ for $i = 1, 2, \ldots, r$ and (ii) S is independent, i.e. contains no edges of G.
Solution: We can remove vertices from each V_i if needed and so we can assume w.l.o.g. that $|V_i| = 10d$ for $i = 1, 2, \ldots, r$. Choose v_i randomly from V_i for $i = 1, 2, \ldots, r$ and let $S = \{v_1, v_2, \ldots, v_r\}$. For an edge $e = \{x, y\} \in E$ we let $\mathcal{E}_e$ be the event that both $x, y \in S$. Thus $\mathbf{P}(\mathcal{E}_e) \leq p = \frac{1}{100d^2}$. An event $\mathcal{E}_e$ depends only on events $\mathcal{E}_f$ for which e and f shae a common vertex. Thus the dependency graph has degree at most $20d^2$. So, $4dp \leq \frac{80d^2}{100d^2} < 1$.

Q3: (40pts)
(a) Show that in any 3-coloring of the edges of K_{17} there is a monochromatic triangle.
Solution: Vertex 1 has degree 16 and so at least one color is used at least 6 times. Let this color be red and suppose that $\{1, j\}$ is red for $j = 2, 3, \ldots, 7$. If there is a red edge $\{x, y\}$ in $[2, 7]$ then $\{1, x, y\}$ is a red triangle. Otherwise $[2, 7]$ is 2-colored and because $R(3, 3) = 6$ it must contain a monochromatic triangle.
(b) Give a 3-coloring of the edges of K_{10} without a monochromatic triangle.
Solution: Partition $[10]$ into two 5-sets S_1, S_2. Color the edges between S_1, S_2 with color 1. This does not create a triangle of color 1. Then because $R(3, 3) = 6$, we can use colors 2,3 to color the edges in each V_i without creating a monochromatic triangle.