Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2015: Test 1

Name:____________________________________

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Problem	Points	Score
1 (a)	20	
1 (b)	20	
2	40	
3	20	
Total	100	

Q1: (40pts)
How many sequences $a_1, a_2, \ldots, a_m \in [n]$ satisfy $a_{i+1} - a_i \geq 2$ for $1 \leq i < m$?
Solution: Let $a_1 = 0$ and $a_{m+1} = n$ and $b_i = a_{i+1} - a_i$, $i = 1, 2, \ldots, m$. Then we have

$$b_0 + b_1 + \cdots + b_m = n \text{ and } b_0 \geq 1, b_m \geq 0, b_i \geq 2, i = 2, 3, \ldots, m-1.$$

There is a 1-1 correspondence between the two sequences $(a_i), (b_i)$ and the number of choices for the b_i's is

$$\binom{m+n-1-2(m-1)-1}{m+1-1} = \binom{n-m}{m}.$$

Q2: (40pts)
Let $G = (V, E)$ be a graph with $|V| = n$ and minimum degree at least $\delta \geq 10$. Show that there is a set A of size at most $\frac{2\log\delta+1}{\delta}n$ such that each vertex not in A is adjacent to at least two vertices in A.
Solution: Choose S_1 from $[n]$ by inclding each element independently with probability p. Then let S_2 be the set of elements, notin S_1, with at most one neighbor in S_1. Let $S = S_1 \cup S_2$. It satisfies the requirements of A. Then,

$$\mathbf{E}(|S|) \leq np + n(1-p)((1-p)^{\delta} + \delta p(1-p)^{\delta-1}) \leq np + ne^{-\delta p}(1 + \delta p).$$

Putting $p = \frac{\log\delta}{\delta}$ gives

$$\mathbf{E}(|S|) \leq n\frac{\log\delta}{\delta} + \frac{n}{\delta}(1 + \delta p) = \frac{2\log\delta + 1}{\delta}n.$$

This implies the existence of A.

Q3: (20pts)
A set A of $\{0,1\}$ strings of length n is said to be (n,k)**-universal** if for any subset of k coordinates $S = \{i_1, i_2, \ldots, i_k\}$ the projection

$$A_S = \{(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) : \ (a_1, a_2, \ldots, a_n) \in A\}$$

contains all possible 2^k strings of length k. Show that if $\binom{n}{k} 2^k (1 - 2^{-k})^r < 1$ then there is an (n,k)-universal set of size r.
Solution: Let A be a set or r random strings. If A is not (n,k)-universal then there exists a set S of size k and an $x \in \{0,1\}^k$ that is not the projection of a string in A. Let $\mathcal{E}_{S,x}$ denote this event. The union bound implies that the probability of this is at most

$$\sum_S \sum_x \mathbf{P}(\mathcal{E}_{S,x}) = \binom{n}{k} 2^k \left(1 - \frac{1}{2^k}\right)^r < 1.$$

This implies the existence of an (n,k)-universal set of size r. by assumption.