Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2012: Test 3

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Write your name and Andrew ID on every page.

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
In a sequence $a_1, a_2, \ldots, a_n$ of real numbers, let p be the maximum number of times a value is repeated, let q be the maximum length of a strictly monotone increasing sub-sequence and let r be the maximum length of a strictly monotone decreasing sub-sequence. Prove that $pqr \geq n$.
A subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ is strictly monotone increasing (resp. decreasing) if $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$ (resp. $a_{i_1} > a_{i_2} > \cdots > a_{i_k}$).
Answer: By the pigeon hole principle there must be at least $\lceil n/p \rceil$ distinct values in the sequence. Prune the sequence to remove repetitions. By the Erdős-Szekerés theorem, as applied to this sub-sequence, we have $qr \geq \lceil n/p \rceil$.

Q2: (40pts)
Let $R(p,q,r)$ denote the minimum n such that if the edges of K_n are colored Red, Blue or Green , then there exists a Red K_p or a Blue K_q or a Green K_r (inclusive or).

(a) Prove that

$$R(p,q,r) \leq R(p-1,q,r) + R(p,q-1,r) + R(p,q,r-1).$$

(b) Deduce from (a) that $R(p,q,r) \leq 3^{p+q+r}$.

Answer:

(a) Let $n = R(p-1,q,r)+R(p,q-1,r)+R(p,q,r-1)$. We apply induction on $p+q+r$. The base case $p=q=1$ is trivial.

Consider the coloring of the edges incident with vertex 1. Let V_R, V_B, V_G be the neighbors of 1 whose edges to 1 are colored Red, Blue, Green respectively. Then, $|V_R| \geq R(p-1,q,r)$ or $|V_B| \geq R(p,q-1,r)$ or $|V_G| \geq R(p,q,r-1)$. Assume without loss that $|V_R| \geq R(p-1,q,r)$. Consider the coloring of the edges contained in V_R. Either there is a Red K_{p-1} which with 1 makes a Red K_p or there is a Blue K_q or a Green K_r.

(b) It follows by induction on $p+q+r$ that

$$R(p,q,r) \leq 3^{p-1+q+r} + 3^{p+q-1+r} + 3^{p+q+r-1} = 3^{p+q+r}.$$

Q3: (20pts)
Let R_k denote the minimum value of n such that if the edges of K_n are colored with k colors then there must be a monochromatic path of length two. Prove that

$$R_k \leq \begin{cases} k+1 & k \text{ even} \\ k+2 & k \text{ odd} \end{cases}$$

Answer: Suppose that we color K_n with k colors and $n \geq k+2$ then vertex 1 is incident with more than k edges and so two edges incident with vertex 1 are of the same color. This implies that there is a monochromatic path. So, $R_k \leq k+2$.
Suppose now that k is even and $n = k+1$. Then each vertex has degree k and to avoid a monochromatic path, each vertex is incident to a vertex of each color and edges of the same color form a matching. In particular, every vertex is incident with an edge of color 1. But, since $n = k+1$ is odd, there are no perfect matchings and so $R_k \leq k+1$.