Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2012: Test 2

Name:____________________________________

Andrew ID:____________________________________

Write your name and Andrew ID on every page.

Problem	Points	Score
1 (a)	20	
1 (b)	20	
2	40	
3	20	
Total	100	

Q1: (40pts)

The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1, a_1 = 9$ and

$$a_n = 6a_{n-1} - 9a_{n-2}$$

for $n \geq 2$.

(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

(b): Find an expression for a_n, $n \geq 0$.

Answer:

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (6a_{n-1} - 9a_{n-2})\, x^n$$

$$a(x) - 9x - 1 = 6x(a(x) - 1) - 9x^2 a(x)$$

$$a(x) = \frac{3x+1}{9x^2 - 6x + 1} = \frac{3x+1}{(1-3x)^2}$$

$$a(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} (3x)^n + 3x \sum_{n=0}^{\infty} \binom{n+1}{1} (3x)^n$$

$$a_n = (n+1)3^n + n3^n = 3^n + 2n3^n$$

Q2: (40pts)
Recall that a *tournament* is an oriented complete graph. A *Hamiltonian path* in a tournament is a directed path that visits every vertex exactly once. Show that there is a tournament of size n with $n!2^{-(n-1)}$ Hamiltonian paths.
Answer: Consider a random tournament T and let X be the number of Hamiltonian paths in T. For a permutation $\sigma\colon [n] \to [n]$ define X_σ by

$$X_\sigma = \begin{cases} 1 & \sigma \text{ gives a Hamiltonian path. I.e., } ((\sigma(i), \sigma(i+1)) \in T, \forall i \in [n]\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = \sum_\sigma X_\sigma$. Clearly

$$\Pr[X_\sigma] = 2^{-(n-1)}$$

and thus $\mathrm{E}_T[X] = n!2^{-(n-1)}$. In particular there is such a tournament.

Q3: (20pts)
A *cover* in a graph is a set of vertices C such that every vertex not in C has a neighbour in C. Show that every graph on n vertices with minimal degree δ has a cover of size at most $n(1 + \ln(\delta + 1))/(\delta + 1)$.
Answer: We follow the same ideas as in finding a cover for the hypercube (the one we used to find a solution for the Hat problem, see slides $20 - 22$ in the presentation "The Probabilistic Method").
We pick a set of vertices at random, where every vertex has probability p of being in the set. Call the set X. Let Y be the set of vertices that are not covered by X (i.e. they are not in X and do not have a neighbor in X). Clearly $X \cup Y$ is a cover and

$$\mathrm{E}[|X \cup Y|] = \mathrm{E}[|X| + |Y|] = \mathrm{E}[|X|] + \mathrm{E}[|Y|] = np + n(1-p)^{\delta+1}.$$

So there is a cover of size at most $n(p + (1-p)^{\delta+1}) \le n(p + e^{-p(\delta+1)})$.
The last stage is optimization — finding the probability p that will minimize the size of the cover:

$$\left(p + e^{-p(\delta+1)}\right)' = 1 - e^{-p(\delta+1)}(\delta+1) = 0$$
$$\Downarrow$$
$$p = \frac{\ln(\delta+1)}{\delta+1}.$$

Plugging p back in $n(p + e^{-p(\delta+1)})$ we get that the expected cover size is at most

$$n\left(\frac{\ln(\delta+1)+1}{\delta+1}\right).$$

Therefore there is a cover of that size in the graph.