Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2012: Test 1

Name:____________________________________

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Problem	Points	Score
1 (a)	20	
1 (b)	20	
2	40	
3	20	
Total	100	

Q1: (40pts)
Let n, s be positive integers.

(a) Prove:

$$\sum_{k=1}^{n} \binom{n}{k} \binom{s-1}{k-1} = \binom{n+s-1}{n-1}.$$

Answer:

$$\sum_{k=1}^{n} \binom{n}{k} \binom{s-1}{k-1} = \sum_{k=1}^{n} \binom{n}{n-k} \binom{s-1}{k-1} = \binom{n+s-1}{n-1},$$

where the second equality is (a variant of) Vandermonde's identity — the number of ways to choose a subset of size $n-1$ out of a set of size $n+s-1$ is the same as the number of ways to choose a subset of size $n-k$ (for $1 \leq k \leq n$) out of the first n elements, and another $k-1$ out of the remaining $s-1$.

(b) Find a closed form expression without a sum for

$$\sum_{k=1}^{n} \binom{n}{k} \binom{s}{k} k.$$

Answer:

$$\sum_{k=1}^{n} \binom{n}{k} \binom{s}{k} k = \sum_{k=1}^{n} \binom{n}{k} \binom{s-1}{k-1} s = s\binom{n+s-1}{n-1}.$$

Q2: (40pts)
How many strings in $\{0,1\}^n$ have k ones, $n-k$ zeros and at least i zeros between the i'th one and the $(i+1)$'st one (for $1 \leq i \leq k-1$)?
Answer: This is like the famous "pirates and gold" with $k+1$ pirates, $n-k$ coins and where the i'th pirate gets at least i coins (for all $0 \leq i \leq k-1$). Hence we first give $\sum_{i=0}^{k-1} i = \binom{k}{2}$ coins, then spread the other with no other conditions. We can do that in

$$\binom{n-k-\binom{k}{2}+k}{k} = \binom{n-\binom{k}{2}}{k}$$

ways.

Q3: (20pts) n families, each consists of a man, a woman and a child are meeting in a restaurant. In how many way can they sit around a table such that:

- Every child sits between a man and a woman, and every man sits to the left of a woman so that around the table the seats are filled M,C,W,M,C,W,...
- No three consecutive seats are filled by the same family.

Answer: Let d_k be the number of ways to put k ones and $3n-k$ zeros on the vertices of a polygon with $3n$ vertices such that there are at least two zeros between each one. Then

$$d_k = \frac{3n}{k}\binom{3n-k-2k+k-1}{k-1} = \frac{3n}{k}\binom{3n-2k-1}{k-1} \qquad ; \qquad d_0 = 1.$$

For brevity we write

$$d_k = \frac{3n}{3n-2k}\binom{3n-2k}{k}.$$

Let A_i be the set of sittings in which the i'th family sits together, and for any $S \subseteq [n]$ let $A_S = \bigcap_{i\in S} A_i$. Then,

$$|A_S| = 3 \cdot d_{|S|} \cdot (|S|)! \cdot (n-|S|)!^3,$$

unless $n = |S| = 1$ in which case $|A_{\{1\}}| = 3$ (this is the only case in which the cuclic symmetry plays another role. Check!).
There are three sets of chairs (according to their number mod three) and we need to choose who gets to sit on mod zero seats (3 options, men, women or children). Notice that going clockwise we will always see the pattern $MCWMCWMCWMCW$. Then we have $d_{|S|}$ options to choose seats for the families in S, then $|S|!$ ways of putting them in their special seats. and $(n-|S|)!^3$ for all the rest.
Therefore, by the inclusion—exclusion principle, there are

$$\sum_{S\subseteq[n]} (-1)^{|S|}|A_S| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3d_k k!(n-k)!^3.$$

sittings when $n > 1$. There are none when $n = 1$.