Department of Mathematics
Carnegie Mellon University

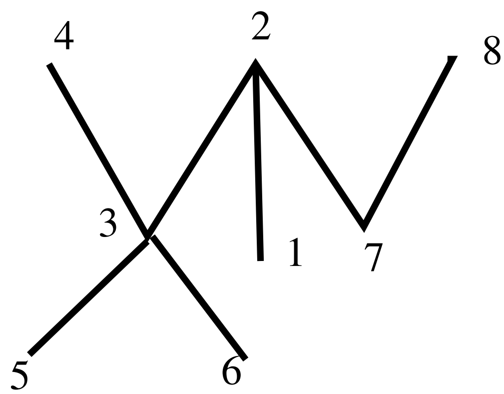
21-301 Combinatorics, Fall 2011: Test 4

Name:__________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)

(a) What is the Prüfer code of the following tree:

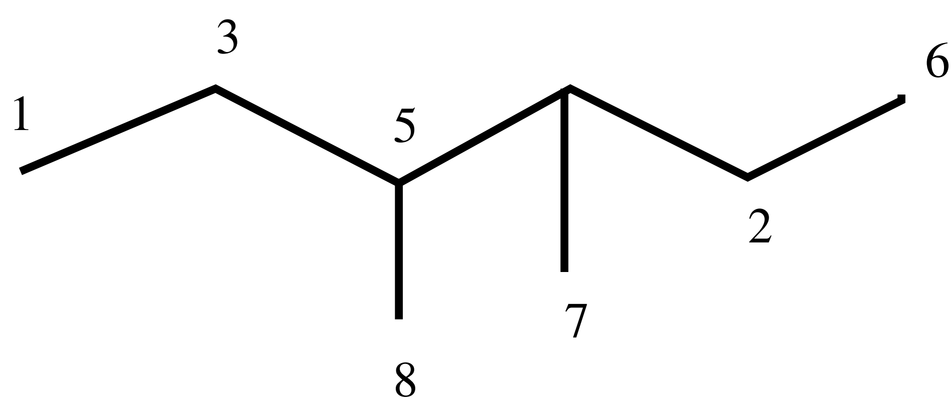


(b) Which tree on 8 vertices has Prüfer code 3,5,2,4,4,5?

(c) How many trees with vertex set $[n]$ that have a vertex of degree $m = \lceil 3n/4 \rceil$?

Solution:

(a) 2,3,3,3,2,7



(b)

(c) There will be $n\binom{n-2}{m-1}(n-1)^{n-1-m}$ such trees.
(Number of choices for vertex (n) times number of choices for the $m-1$ positions in the code times number of choices for rest of the code).

Q2: (40pts)
Consider the following take-away game: There is a pile of n chips. A move consists of removing 2 or 4 chips. Determine the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ and prove that they are what you claim.
Solution: After looking at the first few numbers $0, 0, 1, 1, 2, 2, 0, 0, 1, 1, 2, 2, 0, \ldots$ one sees that

$$g(n) = \begin{cases} 0 & n = 0, 1 \mod 6 \\ 1 & n = 2, 3 \mod 6 \\ 2 & n = 4, 5 \mod 6 \end{cases}$$

We verify this by induction. It is true for $n \leq 12$ by inspection. For $n > 12$ we have that if $n = 6m + s$ then

$$g(n) = mex\{g(n-4), g(n-2)\} = mex\{g(6(m-1)+s+2), g(6(m-1)+s+4)\}$$

So, by induction

$$g(n) = \begin{cases} mex\{g(6(m-1)+2), g(6(m-1)+4)\} = mex\{1, 2\} = 0 & s = 0 \\ mex\{g(6(m-1)+3), g(6(m-1)+5)\} = mex\{1, 2\} = 0 & s = 1 \\ mex\{g(6(m-1)+4), g(6m+1)\} = mex\{2, 0\} = 1 & s = 2 \\ mex\{g(6(m-1)+5), g(6m+1)\} = mex\{2, 0\} = 1 & s = 3 \\ mex\{g(6m), g(6m+2)\} = mex\{0, 1\} = 2 & s = 4 \\ mex\{g(6m+1), g(6m+3)\} = mex\{0, 1\} = 2 & s = 5 \end{cases}$$

The result follows by induction.

Q3: (20pts) A staircase of n steps contains coins on some of the steps. Let $(x_1, x_2, \ldots, x_n)$ denote the position with x_j coins on step $j, j = 1, \ldots, n$. A move of *staircase nim* consists of moving any positive number of coins from any step, j, to the next lower step, $j - 1$. Coins reaching the ground (step 0) are removed from play. A move taking, say, x chips from step j, where $1 \le x \le x_j$, and putting them on step $j - 1$, leaves $x_j - x$ coins on step j and results in $x_{j-1} + x$ coins on step $j - 1$. The game ends when all coins are on the ground. Players alternate moves and the last to move wins. Show that $(x_1, x_2, \ldots, x_n)$ is a P-position if and only if the numbers of coins on the odd numbered steps, $(x_1, x_3, \ldots, x_k)$ where $k = n$ if n is odd and $k = n - 1$ if n is even, forms a P-position in ordinary nim.
Solution: The key is to realise that the coins on the even steps do not matter. If $x_1, x_3, \ldots, ...$ is a Nim N-position then one can move some coins from an odd step and make a Nim P-position. If $x_1, x_3, \ldots, ...$ is a Nim P-position and the coins are moved from an odd step then the coins on the odd steps now form a Nim N-position. If $x_1, x_3, \ldots, ...$ is a Nim P-position and x coins are moved from step $2j$ to step $2j - 1$ then this creates a Nim P-position in $x_1, x_3, \ldots, ...$ which can be countered by moving the x coins to step $2j - 2$.