Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2011: Test 3

Name:______________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
(a) Show that a graph G with minimum degree at least δ contains a path of length δ.
(b) Suppose that the edges of $K_n, n = s + t$ are colored red or blue. Show that either (i) there is a vertex incident to s red edges or (ii) there is a blue path of length t.
Solution:
(a) Let $P = (x_0, x_1, \ldots, x_k)$ be a longest path in G. Since P is a longest path, all neighbors of x_0 are contained in $\{x_1, x_2, \ldots, x_k\}$. The degree of x_0 is at least δ and so $k \geq \delta$.
(b) If there is no vertex with blue degree s or more, then every vertex has red degree at least $s + t - 1 - (s - 1) = t$ and we can just apply part (a).

Q2: (40pts)
Let n, p, q be positive integers and let $N = pqn^2+1$. Suppose that $x_1, x_2, \ldots, x_N$ are positive integers. Show that either (i) there is a subsequence of length $n+1$ in which each successive term increases by a multliple of q or (ii) a subsequence of length $n+1$ in which each successive term decreases by a multiple of q or (iii) a *constant* subsequence of length $p+1$.
Solution: If (iii) does not hold then there are at least $\lceil N/p \rceil = qn^2 + 1$ distinct values in the sequence. For $0 \leq i < q$ let σ_i be the sub-sequence consisting of distinct values equal to $i \mod q$. There must be some i such that σ_i is of length at least $n^2 + 1$. We can then apply the Erdős-Szekerés theorem to this sub-sequence.

Q3: (20pts) Let $\Omega = \Sigma^n$ denote the set of sequences over the alphabet $\Sigma = \{a, b, c\}$. We say that sequences $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$ **collide** if there exists $1 \leq i \leq n$ such that $x_i = y_i$. $A \subseteq \Omega$ is a **collidng set** if every pair $x, y \in A$ collide. Determine a value L such that
(i) $|A| \leq L$ for every colliding set A.
(ii) There exists a colliding set of size L.
Solution:
(i) We first define the function $f : \{a, b, c\} \to \{a, b, c\}$ by $f(a) = b, f(b) = c, f(c) = a$. For $x = x_1 x_2 \cdots x_n \in \Omega$ we define
$f(x) = f(x_1) f(x_2) \cdots f(x_n) \in \Omega$. Note that $x, f(x), f^2(x)$ are distinct and do not collide and also $f^3(x) = x$. Let $A(x) = \{x, f(x), f^2(x)\}$. Note next that if $y \notin A(x)$ then $A(y) \cap A(x) = \emptyset$. The sets $A(x), x \in \Omega$ partition Ω and each colliding set contains at most one sequence in each $A(x)$ and so a colliding set has size at most $|\Omega|/3 = 3^{n-1}$.
(ii) The set of sequences with $x_1 = a$ is a colliding set of size 3^{n-1}.