Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 2

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following:
$a_0 = 1$ and

$$a_n - 7a_{n-1} = 6^n$$

for $n \geq 1$.
(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
(b): Find an expression for a_n, $n \geq 0$.

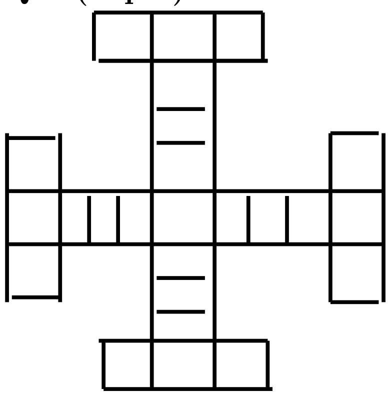
[Hint: $\frac{1}{(1-7x)(1-6x)} = \frac{7}{1-7x} - \frac{6}{1-6x}$.]
Solution: Multiply each equation by x^n and sum.
We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 7a_{n-1})x^n &= \sum_{n=1}^{\infty} 6^n x^n. \\
(a(x) - 1) - 7xa(x) &= \frac{1}{1-6x} - 1. \\
a(x) &= \frac{1}{(1-6x)(1-7x)} \\
&= \frac{7}{1-7x} - \frac{6}{1-6x} \\
&= \sum_{n=0}^{\infty}(7^{n+1} - 6^{n+1})x^n.
\end{aligned}$$

So,

$$a_n = 7^{n+1} - 6^{n+1}$$

Q2: (40pts)



How many ways are there of k-coloring the squares of the above picture if the group acting is $e_0, e_1, e_2, e_3, p, q, r, s$ where e_j is rotation by $2\pi j/4$ and p, q, r, s are horizontal, vertical and diagonal reflections respectively.

Each arm has $n+2$ squares, so that there are $4n+5$ squares altogether.
(All small squares are meant to be of the same size here).
Solution:

g	e_0	e_1	e_2	e_3	p	q	r	s
$\|Fix(g)\|$	k^{4n+5}	k^{n+2}	k^{2n+3}	k^{n+2}	k^{3n+2}	k^{3n+2}	k^{2n+3}	k^{2n+3}

So the total number of colorings is

$$\frac{k^{4n+5} + k^{n+2} + k^{2n+3} + k^{n+2} + k^{3n+2} + k^{3n+2} + k^{2n+3} + k^{2n+3}}{8}.$$

Q3: (20pts) Let G be a bipartite graph with n vertices in total. Each vertex v is given a set $C(v) \subseteq C$ of possible colors. Suppose that $|C(v)| > \log_2 n$. Show that it is possible to assign a color $c(v) \in C(v)$ for each $v \in V$ such that the coloring is proper. I.e. each edge of G joins vertices with different colors.

Solution: Let $G = (V, E)$ where $V = A \cup B$ and all edges have one end in A and the other in B. Randomly partition C into $C_A \cup C_B$. If the event

$$\mathcal{E} = \{C(v) \cap C_A \neq \emptyset \forall v \in A \text{ and } C(v) \cap C_B \neq \emptyset \forall v \in B\}$$

occurs then we get a proper coloring by coloring $v \in A$ with a color in $C(v) \cap C_A$ and a vertex in B with a color in $C(v) \cap C_B$.

Now

$$\neg\mathcal{E} = \{\exists v \in A : C(v) \subseteq C_B\} \cup \{\exists v \in B : C(v) \subseteq C_A\}$$

and so

$$\Pr(\neg\mathcal{E}) \leq \sum_{v \in V} \frac{1}{2^{|C)v)|}} < \frac{n}{2^{\log_2 n}} = 1$$

and so $\mathcal{E}$ occurs with positive probability and we are done.