# Department of Mathematics
# Carnegie Mellon University

## 21-301 Combinatorics, Fall 2010: Test 1

Name:____________________________________

| Problem | Points | Score |
|---|---|---|
| 1 | 40 | |
| 2 | 40 | |
| 3 | 20 | |
| Total | 100 | |

**Q1: (40pts)**
Prove that for $1 \le k \le n$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k}.$$

[Hint: Use induction on $k$.]
**Solution:** When $k = 1$ we have

$$\binom{n}{0} - \binom{n}{1} = -(n-1) = -\binom{n-1}{1}$$

and the inductive hypothesis holds for $k = 1$.
In general

$$\begin{aligned}
&\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^k \binom{n}{k} + (-1)^{k+1} \binom{n}{k+1} = \\
&(-1)^k \binom{n-1}{k} + (-1)^{k+1} \binom{n}{k+1} \qquad \text{using the induction hypothesis} \\
&= (-1)^{k+1} \left( \binom{n}{k+1} - \binom{n-1}{k} \right) \\
&= (-1)^{k+1} \binom{n-1}{k+1} \qquad \text{using Pascal triangle identity.}
\end{aligned}$$

**Q2: (40pts)**
Prove that there are $\frac{n}{k}\binom{n-2k-1}{k-1}$ ways of labelling the vertices of an $n$-cycle with $n-k$ 0's and $k$ 1's so that each 1 is separated by at least 2 0's.
**Solution:** Choose a vertex of the cycle in $n$ ways and then choose the spaces between the remaining $k-1$ 1's as $a_1, a_2, \ldots, a_k$ where $a_1+a_2+\cdots+a_k = n-k$ and $a_1, a_2, \ldots, a_k \geq 2$. There are $\binom{n-k-2k+k-1}{k-1}$ ways of choosing the $a_i$'s and each possible labelling of the cycle appears exactly $k$ times this way.

**Q3: (20pts)** How many solutions are there to

$$x_1 + x_2 + \cdots + x_n = m$$
$$x_j \in \{0, 1, 2, \ldots, B\} \text{ for } j = 1, 2, \ldots, n.$$

Justify your answer.

**Solution:** Let $A$ be the set of all non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = m$ and then let

$$A_i = \{\mathbf{x} \in A : x_i \geq B + 1\}$$

for $i = 1, 2, \ldots, n$.

We want $\left|\bigcap_{i=1}^{n} \bar{A}_i\right|$. Now if $|S| = s$ then

$$|A_S| = |\{\mathbf{y} :\ y_1 + y_2 + \cdots + y_n = m - (B+1)s\} = \binom{m - (B+1)s + n - 1}{n-1}.$$

So the answer is

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{m - (B+1)s + n - 1}{n-1}.$$