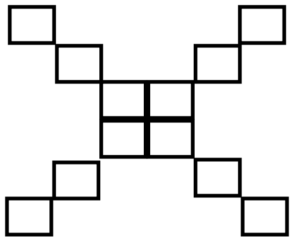
# Department of Mathematics
# Carnegie Mellon University

## 21-301 Combinatorics, Fall 2010: Test 4

Name:____________________________________

| Problem | Points | Score |
|---|---|---|
| 1 | 40 | |
| 2 | 40 | |
| 3 | 20 | |
| Total | 100 | |

**Q1: (40pts)**



How many ways are there of $k$-coloring the squares of the above picture if the group acting is $e_0, e_1, e_2, e_3, p, q, r, s$ where $e_j$ is rotation by $2\pi j/4$ and $p, q, r, s$ are horizontal, vertical and diagonal reflections respectively.
(All small squares are meant to be of the same size here).

**Solution**

| $g$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|---|---|---|---|
| $\|Fix(g)\|$ | $k^{12}$ | $k^3$ | $k^6$ | $k^3$ | $k^6$ | $k^6$ | $k^9$ | $k^9$ |

So the total number of colorings is

$$\frac{k^{12} + k^3 + k^6 + k^3 + k^6 + k^6 + k^9 + k^9}{8}.$$

**Q2: (40pts)**
Consider the following take-away game: There is a pile of $n$ chips. A move consists of removing 2 or 3 chips. Determine the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ and prove that they are what you claim.
**Solution:** After looking at the first few numbers $0, 0, 1, 1, 2, 0, 0, 1, 1, 2, \ldots$ one sees that

$$g(n) = \begin{cases} 0 & n = 0, 1 \mod 5 \\ 1 & n = 2, 3 \mod 5 \\ 2 & n = 4 \mod 5 \end{cases}$$

We verify this by induction. It is true for $n \leq 10$ by inspection. For $n > 10$ we have that if $n = 5m + s$ then

$$g(n) = mex\{g(n-3), g(n-2)\} = mex\{g(5(m-1)+s+2), g(5(m-1)+s+3)\}$$

So, by induction

$$g(n) = \begin{cases} mex\{g(5(m-1)+2), g(5(m-1)+3)\} = mex\{1, 1\} = 0 & s = 0 \\ mex\{g(5(m-1)+3), g(5(m-1)+4)\} = mex\{1, 2\} = 0 & s = 1 \\ mex\{g(5(m-1)+4), g(5m)\} = mex\{2, 0\} = 1 & s = 2 \\ mex\{g(5m), g(5m+1)\} = mex\{0, 0\} = 1 & s = 3 \\ mex\{g(5m+1), g(5m+2)\} = mex\{0, 1\} = 2 & s = 4 \end{cases}$$

The result follows by induction.

**Q3: (20pts)**

**(a)** Let $G = (X \cup Y, E)$ be a bipartite graph with bipartition $X, Y$. A (2,1)-matching is a set of edges $M$ such that each $x \in X$ is incident with two edges of $M$ and each $y \in Y$ is incident with at most one edge of $M$. Show that $G$ contains a (2,1)-matching iff $|N(S)| \geq 2|S|$ for all $S \subseteq X$.

**(b)** $A$ and $B$ play the following game: $W_1, W_2, \ldots, W_m$ are subsets of $W$. $A$ starts and they take turns coloring the elements of $W$. $A$ uses Red and $B$ uses Blue. $A$ wins if he manages to color all the elements of one of the $W_i$'s Red. $B$ wins if she prevents this.

$|W_i| \geq a$ for $i = 1, 2, \ldots, m$ and each $w \in W$ is in at most $b$ of these sets and $2b \leq a$. Show that $B$ can win.

**Solution:**

**(a)** It follows from Hall's Theorem that $G$ contains a matching $M_1$ that covers $X$. Let $G_1 = (X \cup Y, E \setminus M_1)$. Then if $N_1(S)$ denotes the neighborhood of $S \subseteq X$ in $G_1$,

$$|N_1(S)| \geq |N(S)| - |S| \geq |S|$$

since we delete one edge incident with each $x \in X$. Re-applying Hall's theorem we see that $G_2$ contains a matching $M_2$ that covers $X$ and then $M_1 \cup M_2$ is a (2,1)-matching in $G$.

**(b)** Let $G$ be the bipartite graph with $X = [m]$ and $Y = W$ and $E = \{(i, w) : w \in W_i\}$. For $S \subseteq X$ let $e_S$ denote the number of edges of $G$ that are incident with $S$. Then we have

$$a|S| \leq e_S \leq b|N(S)|$$

and so $|N(S)| \geq 2|S|$. Applying (a) we see that $G$ contains a (2,1)-matching. So, for each $i$ there is a set $X_i \subseteq W_i$, $|X_i| = 2$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$.

$B$ plays as follows: If $A$ colors $x$ Red and $x \in X_i = \{x, y\}$ then she immediately colors $y$ Blue. Otherwise, she can color an arbitrary element Blue. In this way, $A$ cannot color a whole $W_i$ Red, because he cannot color a whole $X_i$ red.