Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2010: Test 3

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
Let $\pi(1), \pi(2), \ldots, \pi(n)$ be an arbitrary permutation of $[n]$. Show that there exists i such that

$$\pi(i) + \pi(i+1) + \pi(i+2) \geq \left\lceil \frac{3(n+1)}{2} \right\rceil. \qquad (1)$$

(Here $\pi(n+1) = \pi(1)$ etc.).
(Hint: Consider the sum of LHS(1) over i).
Solution: Let $L_i = \pi(i) + \pi(i+1) + \pi(i+2)$. Then

$$\sum_{i=1}^{n} L_i = 3(1 + 2 + \cdots + n) = 3\frac{n(n+1)}{2}.$$

So there must exist i such that $L(i) \geq 3(n+1)/2$ and we can round this up.

Q2: (40pts)
Let G_1 be a fixed graph. Let $r(G_1, G_1)$ be the smallest integer N such that if we two-color the edges of the complete graph K_N Red or Blue then there is a Red copy of G_1 or a Blue copy of G_1, or both. Show that if P_3 denotes a path of three edges then $r(P_3, P_3) = 5$. Note that a triangle is not a path of three edges.

Solution:

$R(P_3, P_3) > 4$**:** We color edges incident with 1 Red and the remaining edges $\{(2,3),(3,4),(4,1)\}$ Blue. There is no mono-chromatic P_3.

$R(P_3, P_3) \leq 5$**:** There must be two edges of the same color incident with 1. Assume then that $(1,2),(1,3)$ are both Red. If any of $(2,4),(2,5),(3,4),(3,5)$ are Red then we have a Red P_3. If all four of these edges are Blue then $(4,2,5,3)$ is Blue.

Q3: (20pts) Let $\mathcal{A}$ be a subset of $\binom{[n]}{k}$ i.e a collection of k-subsets of $[n]$. Let $\partial\mathcal{A} = \left\{X \in \binom{[n]}{k-1} : \exists Y \in \mathcal{A} \; s.t. \; X \subseteq Y\right\}$. Show that

$$\frac{|\mathcal{A}|}{\binom{n}{k}} \leq \frac{|\partial\mathcal{A}|}{\binom{n}{k-1}}.$$

Solution: For $S \in \binom{[n]}{k-1}$ and $T \in \binom{[n]}{k}$ we let $a(S,T) = 1$ if $S \subseteq T$ and $a(S,T) = 0$ otherwise. Then

$$k|\mathcal{A}| = \sum_{\substack{S\in\partial\mathcal{A} \\ T\in\mathcal{A}}} a(S,T) \leq (n-k+1)|\partial\mathcal{A}|$$

The $=$ follows from the fact that each $T \in \mathcal{A}$ has exactly k sub-sets in $\partial\mathcal{A}$ and the $\leq$ follows from the fact that there are $n-k+1$ k-sets containing any $(k-1)$-set. It follows that

$$\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{k}{n-k+1} = \frac{\binom{n}{k-1}}{\binom{n}{k}}.$$

Here is a really slick solution using the LYM inequality. $\mathcal{A} \cup \left(\binom{[n]}{k-1} \setminus \partial\mathcal{A}\right)$ is a Sperner family. So,

$$\frac{|\mathcal{A}|}{\binom{n}{k}} + \frac{\binom{n}{k-1} - |\partial\mathcal{A}|}{\binom{n}{k-1}} \leq 1.$$

Here is another nice solution: Let π be a random permutation of $[n]$ and for a set X let $\mathcal{E}_X$ denote the event that $X = \{\pi(1), \pi(2), \ldots, \pi_{|X|}\}$. Then

$$\bigcup_{X\in\mathcal{A}} \mathcal{E}_X \subseteq \bigcup_{Y\in\partial\mathcal{A}} \mathcal{E}_Y$$

because if there exists $X \in \mathcal{A}$ such that $\mathcal{E}_X$ occurs, then there exists $Y \in \partial\mathcal{A}$ such that $\mathcal{E}_Y$ occurs.
But, the events $\mathcal{E}_X$ for $X \in \mathcal{A}$ are disjoint as are the events $\mathcal{E}_Y$ for $Y \in \partial\mathcal{A}$. So

$$\frac{|\mathcal{A}|}{\binom{n}{k}} = \Pr\left(\bigcup_{X\in\mathcal{A}} \mathcal{E}_X\right) \leq \Pr\left(\bigcup_{Y\in\partial\mathcal{A}} \mathcal{E}_Y\right) = \frac{|\partial\mathcal{A}|}{\binom{n}{k-1}}.$$