Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 2

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)

(a) If a tree T has n vertices altogether and every vertex is of degree 1 or $k \geq 2$ then how many vertices of degree k does T have?

(b) How many such trees are there on the vertex set $\{1, 2, \ldots, n\}$?

Solution: Let n_1 be the number of vertices of degree one and let n_k be the number of vertices of degree k. Then

$$n_1 + n_k = n \text{ and } n_1 + kn_k = 2n - 2.$$

This gives us

$$n_k = \frac{n-2}{k-1}.$$

The number of such trees will be

$$\binom{n}{n_k}\binom{n-2}{k-1, k-1, \ldots, k-1}.$$

The binomial coefficient gives the number of choices for the vertices of degree k. Having fixed these, the multinomial coefficient gives the number of trees with degree sequence $k, k, \ldots, k, 1, 1, \ldots, 1$ where there are n_k k's.

Q2: (40pts)
A particle does a random walk on $0, 1, 2, \ldots, L$ until it reaches L where it stops. When at 0 it moves immediately to 1. When at $i \geq 1$ it moves to $i-1$ with probability 1/2 and to $i+1$ with probability 1/2.
When at site s it picks up a reward of s before making a move.
Let E_k denote the expected total reward obtained before stopping at L if we started the walk at k. Thus

$$E_0 = E_1 \text{ and } E_L = L.$$

1. Find a recurrence involving the E_k, $0 \leq k \leq L$.
2. Prove inductively that $E_k = E_{k+1} + k(k+1)$ for $0 \leq k < L$.
3. Deduce that $E_0 = L + \frac{L(L-1)(L+1)}{3}$.

Solution:

1. $E_k = k + \frac{E_{k-1}+E_{k+1}}{2}$.
2. Base Case: $E_0 = E_1$
   Assume true for $k-1$:
   Then
   $$E_k = k + \frac{E_{k-1} + E_{k+1}}{2} = k + \frac{E_k + (k-1)k + E_{k+1}}{2}$$
   and so
   $$E_{k+1} = 2k + (k-1)k = k(k+1).$$
3. It follows from 2 and 3 that
   $$E_L = \sum_{i=0}^{L-1} i(i+1) + E_L = \frac{L(L-1)(L+1)}{3} + L.$$

Q3: (20pts) A graph G has a cycle C and it contains a path P of length k joining two vertices of C. Show that G contains a cycle of length at least $\lceil 2k^{1/2} \rceil$.
(You will get most of the points if you can show that there is a cycle of length at least $k^{1/2}$).
Solution: Suppose that C has length ℓ. If $\ell \geq \lceil 2k^{1/2} \rceil$ then we are done. Let the vertices on $P \cap C$ be $x_1, x_2, \ldots, x_l$ where x_1 and x_l are the start and end of P. The x_i are distinct, as P is a path. The path P breaks up into sub-paths P_i, $i = 1, 2, \ldots, l-1$ where P_i goes from x_i to x_{i+1}. Here $l \leq \ell$. Now at least one of these sub-paths must be of length $\frac{k}{l-1} \geq \frac{k}{\ell-1}$. Take one such path x_j say, and extend it to a cycle by adding the longer of the two paths in C from x_j to x_{j+1}. Thus G contains a cycle of length at least

$$\frac{k}{\ell-1} + \frac{\ell}{2}.$$

Now the above expression is minimised at $\ell = 1 + \sqrt{2k}$ giving $\sqrt{2k} + 1/2$ and the result follows, as the cycle length is integer..