Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2010: Test 1

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following:
$a_0 = 1$ and

$$a_n - 6a_{n-1} = 5^n$$

for $n \geq 1$.
(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
(b): Find an expression for a_n, $n \geq 0$.

[Hint: $\frac{1}{(1-6x)(1-5x)} = \frac{6}{1-6x} - \frac{5}{1-5x}$.]
Solution Multiply each equation by x^n and sum.
We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 6a_{n-1})x^n &= \sum_{n=1}^{\infty} 5^n x^n. \\
(a(x) - 1) - 6xa(x) &= \frac{1}{1-5x} - 1. \\
a(x) &= \frac{1}{(1-5x)(1-6x)} \\
&= \frac{6}{1-6x} - \frac{5}{1-5x} \\
&= \sum_{n=0}^{\infty}(6^{n+1} - 5^{n+1})x^n.
\end{aligned}$$

So,

$$a_n = 6^{n+1} - 5^{n+1}$$

Q2: (40pts)
For positive integers m, n let

$$A = \{\mathbf{x} \in \{0, 1, 2, \ldots, \}^n : \ x_1 + x_2 + \cdots + x_n = m\}.$$

Let

$$B = \{\mathbf{x} \in A : \ x_j \neq 1, j = 1, 2, \ldots, n\}.$$

Use inclusion-exclusion to find an expression for $|B|$. Justify your steps.
Recall the inclusion-exclusion formula:

$$\left| \bigcap_{i=1}^{N} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Solution Let $A_i = \{\mathbf{x} \in A : \ x_i = 1\}$. We want $\left| \bigcap_{i=1}^{N} \bar{A}_i \right|$. Now if $|S| = k$ then $|A_S| = |\{\mathbf{x} \in \{0, 1, 2, \ldots, \}^{n-k} : \ x_1 + x_2 + \cdots + x_{n-k} = m - k\}| = \binom{(m-k)+(n-k)-1}{n-k-1}$. So,

$$|B| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m + n - 2k - 1}{n - k - 1}.$$

Q3: (20pts) Two sequences $a_n, n \geq 0$ and $b_n, n \geq 0$ satisfy the equations

$$b_k = \sum_{j=0}^{k} \binom{k}{j} a_j, \qquad k = 0, 1, 2, \ldots$$

Show that

$$a_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} b_j, \qquad k = 0, 1, 2, \ldots$$

Solution Re-write the first equation as

$$\frac{b_k}{k!} = \sum_{j=0}^{k} \frac{a_j}{j!} \frac{1}{(k-j)!}. \tag{1}$$

Let

$$a(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k \text{ and } b(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k.$$

Now multiply equation (1) by x^k and sum for $k = 0$ to ∞. The result is

$$b(x) = a(x) e^x.$$

Thus $a(x) = b(x) e^{-x}$ and the result follows.

Here is a more direct proof:

$$\begin{aligned}
&\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} b_j \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{\ell=0}^{j} \binom{j}{\ell} a_\ell \\
&= \sum_{\ell=0}^{k} (-1)^{k-\ell} a_\ell \sum_{j=\ell}^{k} (-1)^{\ell-j} \binom{k}{j} \binom{j}{\ell} \\
&= \sum_{\ell=0}^{k} (-1)^{k-\ell} a_\ell \sum_{j=\ell}^{k} (-1)^{\ell-j} \binom{k}{\ell} \binom{k-\ell}{j-\ell} \\
&= \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} a_\ell \sum_{j=\ell}^{k} (-1)^{\ell-j} \binom{k-\ell}{j-\ell}
\end{aligned}$$

Now observe that $\sum_{j=\ell}^{k}(-1)^{\ell-j}\binom{k-\ell}{j-\ell} = 0$ unless $k = \ell$.