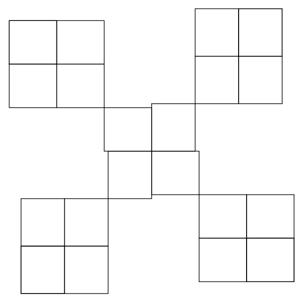
Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 4

Name:________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)



How many ways are there of k-coloring the squares of the above picture if the group acting is $e_0, e_1, e_2, e_3, p, q, r, s$ where e_j is rotation by $2\pi j/4$ and p, q, r, s are horizontal, vertical, diagonal reflections.

Solution:

g	e_0	e_1	e_2	e_3	p	q	r	s
$\lvert Fix(g)\rvert$	k^{20}	k^5	k^{10}	k^5	k^{10}	k^{10}	k^{13}	k^{13}

So the total number of colorings is

$$\frac{k^{20} + k^5 + k^{10} + k^5 + k^{10} + k^{10} + k^{13} + k^{13}}{8}.$$

Q2: (40pts)
Consider the following take-away game: There is a pile of n chips. A move consists of removing 5^k chips for some $k \geq 1$. Compute the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ and prove that what you say is correct.
Suppose now that you are playing a game where there are several piles and a move consists of choosing a pile and then removing 5^k chips for $k \geq 1$ if possible. Suppose that there are 3 piles left of size 13,27,33. Is this a P or N position? If it is an N position, give a possible move.
Solution: After looking at the first few numbers $0, 0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots$ one sees that

$$g(n) = \begin{cases} 0 & n = 0, 1, 2, 3, 4 \mod 10 \\ 1 & n = 5, 6, 7, 8, 9 \mod 10 \end{cases}$$

We verify this by induction. It is true for $n \leq 10$ by inspection. For $n > 6$ we have $g(n) = mex\{g(n-5), g(n-25), \ldots, \}$. Observe that if $k \geq 2$ then $5^k = 5(5^{k-1} - 1) + 5$ and so $5^k \mod 10 = 5$. It follows that $g(n) = mex\{g(n-5)\}$ and the induction step follows.
The Sprague-Grundy number of the position is $g(13) \oplus g(27) \oplus g(33) = 0 \oplus 1 \oplus 0 = 1$ and so it is an N position. Removing 5 or 25 from the second or third piles produces a P position. So does removing 5 from the first pile.

Q3: (20pts) Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1} \in \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$. Show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$

Solution: Let π be a random permutation of $[n]$. Let $\mathcal{E}(A)$ be the event $\{\{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\}$. Let

$$Z_i = \begin{cases} 1 & \mathcal{E}(A_i) \ occurs. \\ 0 & otherwise. \end{cases}$$

and let $Z = \sum_i Z_i$ be the number of events $\mathcal{E}(A_i)$ that occur.
Now our family is such that $Z \leq s$ for all π and so

$$E(Z) = \sum_i E(Z_i) = \sum_i \Pr(\mathcal{E}(A_i)) \leq s.$$

On the other hand, $A \in \mathcal{A}$ implies that $\Pr(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}}$ and the required inequality follows.