Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 3

Name:

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
Prove that if $n > 0$ and $a_1, a_2, \ldots, a_{n+1}$ are distinct positive integers then there is a pair $i < j$ such that $a_i - a_j$ is divisible by n.
Solution:
Consider the $n + 1$ integers $u_k \mod n$, $k = 1, 2, \ldots, n + 1$.

By the PHP there exist k, l such that $u_k = u_l$.

Q2: (40pts)
Let P_3 denote the path a, b, c, d with three edges and four distinct vertices.

(a) Show that the edges of K_4 can be colored red or blue so that there is neither a red copy of P_3 nor a blue copy of P_3.

(b) Show that the edges of K_5 are colored red or blue then there is either a red copy of P_3 or a blue copy of P_3.
(Hint: Concentrate first on the possible colorings of the edges of the 4-cycle C on $\{1, 2, 3, 4\}$.)

Solution:
(a) Color edges $12, 13, 14$ red and edges $23, 24, 34$ blue.
(b) If red is used 3 or 4 times on C then there is a red P_3.
Similarly for blue.
Assume that each color is used twice on C_4.
Either
(i) $12, 13$ are red and $42, 43$ are blue (w.l.o.g.)
If 35 is red then 2135 is red and if 35 is blue then 2435 is blue.
or
(ii) $12, 34$ are red and $13, 24$ are blue (w.l.o.g.)
If 23 is red then 1234 is red and
if 23 is blue then 1324 is blue.

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Q3: (20pts) Re-call that a tournament is a digraph in which for every pair of distinct vertices u, v, **exactly one** of $(u, v), (v, u)$ is an edge.

Show that a tournament on n vertices has a directed path with k edges for every $1 \leq k \leq n - 1$.

(Hint: Use induction on k.)

Solution: The base case $k = 1$ is trivial. 2 points.

Assume there is a path $P = x_1, x_2, \ldots, x_{k+1}$ with k edges where $k < n - 1$.

Choose a vertex y not on P. 2 points.

If the edge joining y, x_1 is oriented y to x_1 then $y x_1, x_2, \ldots, x_{k+1}$ is a path with $k + 1$ edges.

Otherwise, if the edge joining x_{k+1}, y is oriented x_{k+1} to y then $x_1, x_2, \ldots, x_{k+1} y$ is a path with $k + 1$ edges.

Otherwise, y, x_1 is oriented x_1 to y and x_{k+1}, y is oriented y to x_{k+1}.

It follows that there exists $1 \leq \ell \leq k$ such that y, x_ℓ is oriented x_ℓ to y and $x_{\ell+1}, y$ is oriented y to $x_{\ell+1}$.

Then $x_1, \ldots, x_\ell, y, x_{\ell+1}, \ldots, x_{k+1}$ is a path with $k + 1$ edges.