Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 2

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following:
$a_0 = 1$ and

$$a_n - 5a_{n-1} = 4^n$$

for $n \geq 1$.
(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
(b): Find an expression for a_n, $n \geq 0$.

[Hint: $\frac{1}{(1-5x)(1-4x)} = \frac{5}{1-5x} - \frac{4}{1-4x}$.]
Solution: Multiply each equation by x^n and sum.
We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 5a_{n-1})x^n &= \sum_{n=1}^{\infty} 4^n x^n. \\
(a(x) - 1) - 5xa(x) &= \frac{1}{1-4x} - 1. \\
a(x) &= \frac{1}{(1-4x)(1-5x)} \\
&= \frac{5}{1-5x} - \frac{4}{1-4x} \\
&= \sum_{n=0}^{\infty}(5^{n+1} - 4^{n+1})x^n.
\end{aligned}$$

So,

$$a_n = 5^{n+1} - 4^{n+1}$$

Q2: (40pts)
A particle does a random walk on $0, 1, 2, \ldots, L$ until it reaches L where it stops. When at 0 it moves immediately to 1. When at $i \geq 1$ it moves to $i-1$ with probability 1/2 and to $i+1$ with probability 1/2.
Let E_k denote the expected number of visits to 0 before stopping at L if we started the walk at k.

1. What is E_L?
2. Find a recurrence involving the E_k.
3. Prove inductively that $E_k = 1 + E_{k+1}$ for $k < L$.
4. Deduce that $E_k = L - k$ for $k \geq 0$.

Solution:

1. $E_L = 0$
2. $E_k = (E_{k-1} + E_{k+1}/2$
3. Base Case: $E_0 = 1 + E_1$
   Assume true for some k:
   Then $E_{k+1} = ((1 + E_{k+1}) + E_{k+2})/2$ or $E_{k+1} = 1 + E_{k+2}$.

4. This follows by induction from 1 and 3.

Q3: (20pts) Let A be an $n \times n$ real matrix with pair-wise distinct entries. Show that if n is large then there is a permutation of the rows so that no column contains an increasing subsequence of length $10n^{1/2}$.
(If $c_1, c_2, \ldots, c_n$ is a column (after row permutaion), then an increasing subsequence of length t is a sequence $i_1 < i_2 < \cdots < i_t$ such that $c_{i_1} < c_{i_2} < \cdots < c_{i_t}$).
(You can use the inequality $m! \geq (m/e)^m$ if you need to).
Solution: Let π denote a random permutation of the rows.
Let $\mathcal{E}$ denote the event that there is a column with a monotone increasing sequence of size $k = 10n^{1/2}$.
Let $\mathcal{E}(i, S)$ denote the event
$\{S$ entries of column i are monotone increasing$\}$.
$Pr(\mathcal{E}) \leq \sum_{i=1}^{n} \sum_{S} P(\mathcal{E}(i, S)$.
$Pr(\mathcal{E}) \leq n\binom{n}{k}/k! \leq n^{k+1}/(k!)^2 \leq n^{k+1}e^{2k}/k^{2k} \leq n^{k+1}e^{2k}/(100n)^k \to 0.$