Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2009: Test 1

Name:____________________________________

Problem	Points	Score
1	40	
2	40	
3	20	
Total	100	

Q1: (40pts)
k a's, k b's and $n-2k$ c's are placed on the vertices of an n vertex polygon so that each a is followed clockwise by b which is followed by a non-empty sequence of c's until the next a. Show that the number of ways of doing this is $\frac{n}{k}\binom{n-2k-1}{k-1}$.
Solution: There are n ways of choosing a place to put an a. A b follows immediately. Let there be x_i c's between the ith b and the $i+1$'th a. Then we have $x_1+\cdots+x_k=n-2k$ and $x_1,\ldots,x_k\geq 1$. The number of choices for the x's is thus $\binom{n-2k-1}{k-1}$ and we multiply by n/k to account for the choice of the "first" a and for over-counting.

Q2: (40pts)
n children take off their jackets and shoes and put them into a pile on the floor and go and play. How many ways are there of giving to each of the children a pair of matching shoes and a jacket, so that no child gets his/her own jacket and shoes. Your answer should be a summation.
Re-call that if $A_1, A_2, \ldots, A_N \subseteq A$ then

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = \sum_{S\subseteq[N]} (-1)^{|S|}|A_S|.$$

Solution: Suppse that child i is given the jacket of child $\pi_1(i)$ and the shoes of child $\pi_2(i)$. Let

$$A_i = \{(\pi_1, \pi_2) : \ \pi_1(i) = \pi_2(i) = i\}$$

for $i = 1, 2, \ldots, n$.
We need to compute $\left|\bigcap_{i=1}^{n} \bar{A}_i\right|$. Now if $|S| = k$ then $|A_S| = ((n-k)!)^2$ since we have fixed $\pi_1(i), \pi_2(i)$ for $i \in S$ and the remaining values can be permuted arbitrarily. Thus

$$\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = \sum_{S\subseteq[N]} (-1)^{|S|}((n-|S|)!)^2 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}((n-k)!)^2.$$

Q3: (20pts) How many ways are there of placing k a's and $n-k$ b's on the vertices of an n vertex polygon so that each a is seperated by an odd number of b's. There are different answers for n odd or n even.

Solution: There are n choices as to where to put an a on a vertex. Suppose that after this there are x_i b's between each a where $x_1+\cdots+x_k = n-k$ and each x_i is odd. Let d_k be the number of choices for the x's, in which case the solution is nd_k/k.

Each x_i can be written as $2y_i+1$ where $y_i \geq 0$ for $i = 1, 2, \ldots, k$ and $2(y_1+\cdots+y_k)+k = n-k$. If there is a solution then $n = 2(y_1+\cdots+y_k+k)$ is even i.e. there are no solutions for odd n. Otherwise, if n is even then $y_1+\cdots+y_k = n/2-k$ and the number of choices for the y's and hence the x's is $\binom{n/2-k+k-1}{k-1} = \binom{n/2-1}{k-1}$. So the final answer is

$$\frac{n}{k}\binom{n/2-1}{k-1}. \tag{1}$$

Alternate solution found by some students:

The polygon's vertices are partitioned into a sequence of segments where each segment starts with an a and continues with an odd number of b's. Thus each segment is even and so n must be even. Furthermore, if $n = 2m$ and we partition the vertices into $A = \{1, 3, \ldots, 2m-1\}$ and $B = \{2, 4, \ldots, 2m\}$ then all a's must be placed in A or all a's must be placed in b and any such placement is valid. Thus the number of choices is

$$2\binom{n/2}{k}.$$

You can check that this is the same as (1).