Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2008: Test 4

Name:______________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
(a) Show that if you Red-Blue color the edges of K_9 then either there is a vertex with Red degree at least 4 or Blue degree at least 6.
[Hint: Suppose there is no such vertex. Now you know the Red and Blue degrees of each vertex. Now use the fact that the number of odd vertices in a graph is even.]
(b) Show that if you Red-Blue color the edges of K_9 then either there is a Red K_3 or a Blue K_4.
Solution
(a) Let d_R, d_B denote degree in the graphs induced by the Red and Blue edges. If $d_R(v) \leq 3$ and $d_B(v) \leq 5$ for all v then $d_R(v) + d_B(v) = 8$ implies that $d_R(v) = 3$ and $d_B(v) = 5$ for all v. But then the Red graph has an odd number of vertices of odd degree (as does the Blue graph) and this is a contradiction.
(b) Suppose first that there is a vertex of Red degree at least 4. Assume w.l.o.g. that the edges $(1, 2), (1, 3), (1, 4), (1, 5)$ are all Red. Now either the sub-graph spanned by 2,3,4,5 contains a Red edge and so we have a Red K_3. Or the sub-graph spanned by 2,3,4,5 contains only Blue edges and we have a Blue K_4.
Now suppose that there is a vertex of Blue degree at least 6. Assume w.l.o.g. that the edges $(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7)$ are all Blue. Now $R(3, 3) = 6$ and so the sub-graph spanned by 2,3,. . . ,7 contains a Red K_3 or a Blue K_3. In the latter case, adding 1 to the Blue K_3 gives Blue K_4.

Q2: (33pts)
Consider the following two one pile take-away games.
(a): In game one you can take away 2^k chips for any $k \geq 0$. Prove inductively that the Grundy number $g_1(n)$ is given by $g_1(n) = n \mod 3$.
(b): In game two you can take away 3^k chips for any $k \geq 0$. . Prove inductively that the Grundy number $g_2(n)$ is given by $g_2(n) = n \mod 2$.
(c): Now consider the two pile game where one can either take $2^k, k \geq 0$ chips from the first pile or you can take $3^k, k \geq 0$ chips from the second pile. Is the position (150,95) a P or an N position? Justify your claim. If it is an N position, what is a correct move?

Solution

(a) Letting $g_1(0) = 0, g_1(1) = 1, g_1(2) = 2$ and inducting on n we see that for $n \geq 2$

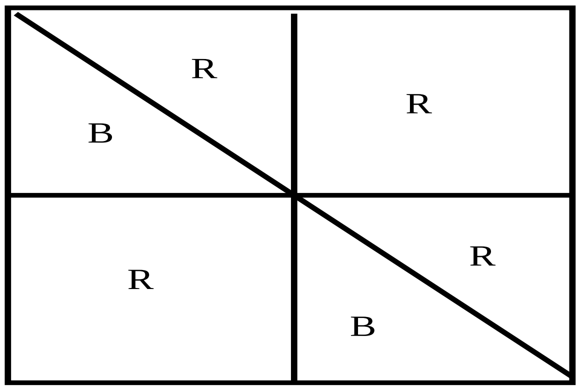
$$\begin{aligned} g_1(n+1) &= \text{mex}\{g_1(n+1-2^k) : \ 0 \leq k \leq \lfloor \log_2(n+1) \rfloor\} \\ &= \text{mex}\{(n+1-2^k \mod 3 : \ 0 \leq k \leq \lfloor \log_2(n+1) \rfloor\} \\ &= \text{mex}\{n \mod 3, n-1 \mod 3\} \\ &= n+1 \mod 3. \end{aligned}$$

(b) Letting $g_2(0) = 0, g_2(1) = 1, g_2(2) = 2$ and inducting on n we see that for $n \geq 1$

$$\begin{aligned} g_2(n+1) &= \text{mex}\{g_2(n+1-3^k) : \ 0 \leq k \leq \lfloor \log_3(n+1) \rfloor\} \\ &= \text{mex}\{(n+1-3^k \mod 2 : \ 0 \leq k \leq \lfloor \log_3(n+1) \rfloor\} \\ &= \text{mex}\{n \mod 2\} \\ &= n+1 \mod 2. \end{aligned}$$

(c) We have $g_1(150) \oplus g_2(95) = 0 \oplus 1 = 1$. So (150,95) is an N position. Taking 2^{2k+1} from pile one or 3^k from pile two is a winning move.

Q3: (34pts) How many distinct Red-Blue colorings D are there of the 6 region diagram below. The group G consists of 2 rotations e, b, ($0°$ and $180°$) and 2 reflections r, s (one on each diagonal).
One possible coloring is shown.



Use the formula

$$D = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|.$$

Solution: Let D denote the number of distinct colorings.

g	e	b	r	s
$\|Fix(g)\|$	2^6	2^3	2^3	2^4

Here e is the identity, a, b, c are rotations and p, q, r, s are reflections.
This gives $D = 24$.