Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2008: Test 3

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Let $G = K_{2,n}$ denote the bipartite graph (X, Y, E) where $|X| = 2, |Y| = n$ and an edge (x, y) for every $x \in X, y \in Y$. A spanning tree T of G is a subgraph of G that (a) is a tree and (b) has vertex set $X \cup Y$. T therefore has $n + 1$ edges.
Show that there are $n2^{n-1}$ spanning trees of $K_{2,n}$.
Solution: A spanning tree of G has the following form: Suppose that $X = \{x_1, x - 2\}$. There is a $y \in Y$ that is adjacent to both x_1, x_2 and every other vertex in Y is adjacent to one of x_1, x_2. (There are $n+1$ edges and each vertex of Y has degree at least 1. Furthermore, every such graph is connected.)
There are n choices for y and 2^{n-1} choices for the set of vertices adjacent to x_1.

Q2: (33pts)
Recall that a tournament $T = ([n], A)$ is an orientation of the complete graph K_n for some n. The out-degree $d^+(x)$ of a vertex x is the number of edges oriented away from x i.e. $|\{y : (x, y) \in A\}|$.

(a) Show that if there is no directed path of length one or two from x to y in T, then $d^+(y) > d^+(x)$.

(b) Deduce that T contains a vertex z such that every vertex is reachable from z by a directed path of length one or two.

Solution:

(a) The edge $\{x, y\}$ must be oriented from y to x. Furthermore, if the edge $\{x, z\}$ is oriented from x to v then the edge $\{y, v\}$ must be oriented from y to v. Otherwise, there is a path x, v, y.

(b) Let z be a vertex of maximum out-degree. For all $x \neq z$, if there was no path of length one or two to x then $d^+(x) > d^+(y)$, contradiction.

Q3: (34pts) Let n, p be positive integers and let $N = 2n^2p + 1$. Suppose that $x_1, x_2, \ldots, x_N$ are positive integers. Show that either (i) there is a *strictly* increasing subsequence of length $n+1$ in which each successive term increases by at least 2 or (ii) a *strictly* decreasing subsequence of length $n+1$ in which each successive term decreases by at least 2 or (iii) a *constant* subsequence of length $p+1$.

Solution: If (iii) does not hold then there must be at least $\lceil N/p \rceil = 2n^2 + 1$ distinct values in the sequence. Thus there are either $\geq n^2 + 1$ distinct odd values or there are $\geq n^2 + 1$ disitnct even values. Assume the former. By Erdős-Szekerés there is a monotone subsequence of distinct odd values of length $\geq n + 1$.