Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2008: Test 2

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following:
$a_0 = 1$ and

$$a_n - 4a_{n-1} = 3^n$$

for $n \geq 1$.
(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
(b): Find an expression for a_n, $n \geq 0$.

[Hint: $\frac{1}{(1-4x)(1-3x)} = \frac{4}{1-4x} - \frac{3}{1-3x}$.]
Solution:
Multiply each equation by x^n and sum. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 4a_{n-1})x^n &= \sum_{n=1}^{\infty} 3^n x^n. \\
(a(x) - 1) - 4xa(x) &= \frac{1}{1-3x} - 1. \\
a(x) &= \frac{1}{(1-3x)(1-4x)} \\
&= \frac{4}{1-4x} - \frac{3}{1-3x} \\
&= \sum_{n=0}^{\infty}(4^{n+1} - 3^{n+1})x^n.
\end{aligned}$$

So,

$$a_n = 4^{n+1} - 3^{n+1}.$$

Q2: (33pts)
Let $A_1, A_2, \ldots, A_n$ be arbitrary subsets of the set A. Suppose that $|A_1| = |A_2| = \cdots = |A_n| = k$. Show

(a) $n < 3^{k-1}$ implies that there exists a partition (coloring) R, B, G of A such that for all i, $A_i \nsubseteq R, A_i \nsubseteq B$ and $A_i \nsubseteq G$ i.e. no A_i is all of one color.

(b) $n < \frac{3^{k-1}}{2^k-1}$ implies that there exists a partition (coloring) R, B, G of A such that for all i, $A_i \cap R \neq \emptyset, A_i \cap B \neq \emptyset$ and $A_i \cap G \neq \emptyset$ i.e. each A_i contains at least one element of each color.

Solution:
(a) Color randomly. Let $\mathcal{A}_i$ be the event that A_i is mono-colored. Then, $\mathbf{P}(\mathcal{A}_i) = 3^{1-k}$ and so

$$\mathbf{P}(\exists i:\ \mathcal{A}_i\ occurs) \leq \sum_{i=1}^{n} \mathbf{P}(\mathcal{A}_i) = \frac{n}{3^{k-1}} < 1.$$

Thus there exists at least one coloring in which no $\mathcal{A}_i$ occurs.
(a) Color randomly. Let $\mathcal{B}_i$ be the event that A_i is not 3-colored. There are $3 \times (2^k - 1)$ ways of coloring A_i so that at most two colors are used. So $\mathbf{P}(\mathcal{B}_i) = \frac{3\times(2^k-1)}{3^k}$.

$$\mathbf{P}(\exists i:\ \mathcal{B}_i\ occurs) \leq \sum_{i=1}^{n} \mathbf{P}(\mathcal{B}_i) = \frac{n(2^k-1)}{3^{k-1}} < 1.$$

Thus there exists at least one coloring in which no $\mathcal{B}_i$ occurs.

Q3: (34pts) A box has 2 drawers. Drawer i contains g_i gold coins and s_i silver coins where $g_i + s_i \geq 1$, for $i = 1, 2$. A person first chooses a random coin c_1 from drawer 1. The person then places this coin in drawer 2. Finally, the person chooses a random drawer d and a random coin from that drawer. What is the probability that this coin is gold?
[Hint: Let A_g be the event $\{c_1 \text{ is gold}\}$, A_s be the event $\{c_1 \text{ is silver}\}$, and B_i be the event $\{d = i\}$ for $i = 1, 2$. Using these events, partition your probability space and use the law of total probability.]
Solution: Let G be the event that the second coin is gold. Then,

$$\mathbf{P}(G) = A + B + C + D$$

where

$$\begin{aligned}
A &= \mathbf{P}(G \mid A_g, D_1)\mathbf{P}(A_g, D_1) = \frac{1}{2} \cdot \frac{g_1 - 1}{g_1 + s_1 - 1} \cdot \frac{g_1}{g_1 + s_1} \\
B &= \mathbf{P}(G \mid A_s, D_1)\mathbf{P}(A_s, D_1) = \frac{1}{2} \cdot \frac{g_1}{g_1 + s_1 - 1} \cdot \frac{s_1}{g_1 + s_1} \\
C &= \mathbf{P}(G \mid A_g, D_2)\mathbf{P}(A_g, D_2) = \frac{1}{2} \cdot \frac{g_2 + 1}{g_2 + s_2 + 1} \cdot \frac{g_1}{g_1 + s_1} \\
D &= \mathbf{P}(G \mid A_s, D_2)\mathbf{P}(A_s, D_2) = \frac{1}{2} \cdot \frac{g_2}{g_2 + s_2 + 1} \cdot \frac{s_1}{g_1 + s_1}
\end{aligned}$$