Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2008: Test 1

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Show that the number of ways of placing k a's and $n-k$ b's or c's on the vertices of an n vertex polygon so that each a is seperated by at least ℓ b's or c's is $\frac{2^{n-k}n}{k}\binom{n-\ell k-1}{k-1}$.
Solution: Let the a's be at position $x_1, x_2, \ldots, x_k$ in order round the polygon. There are n choices for x_1 and then if y_i is the number of b's and c's between x_i and x_{i+1} for $i = 1, 2, \ldots, k$ (here $x_{k+1} = x_1$) then (i) $y_i \geq \ell$, $i = 1, 2, \ldots, k$ and (ii) $y_1 + \ldots + y_k = n - k$. There are $\binom{n-\ell k-1}{k-1}$ choices for the y's and n choices for x_1. There are $n-k$ positions that contain b or c and ther can be filled in in 2^{n-k} ways. Each placement arises exactly k times in this manner.

Q2: (33pts)
In how many ways can n three armed creatures be given a left glove, a right glove and a middle glove from n distinguishable triples of gloves without any creature getting a triple of gloves?
(Hint: Let $A_i = \{Allocations\ of\ gloves\ in\ which\ creature\ i\ gets\ a\ triple\}$.)
Re-call that if $A_1, A_2, \ldots, A_N \subseteq A$ then

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Solution: Fix $S \subseteq [n]$ with $|S| = k$. Then

$$|A_S| = n!(n-k)!^2.$$

Explanation: Choose the allocation of left gloves to creatures in $n!$ ways. The allocations to S are now fixed. The number of allocations of middle and left gloves to creatures not in S is $(n-k)!^2$.
Thus the number of allocations is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} n!(n-k)!^2 = n!^2 \sum_{k=0}^{n} (-1)^k \frac{(n-k)!}{k!}.$$

Q3: (34pts)

(a) Show that
$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

(b) Show that
$$\sum_{k=0}^{m} \binom{n-k}{m-k} = \binom{n+1}{m}.$$

(Hint: How many m-subsets of $[n+1]$ contain $n+1, n, \ldots, n-k+2$ and not $n-k+1$?).

(c) Deduce that
$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{\binom{n+1}{m}}{\binom{n}{m}} = \frac{n+1}{n+1-m}.$$

Solution:

(a)
$$\binom{n}{m}\binom{m}{k} = \frac{n!}{m!(n-m)!}\frac{m!}{k!(m-k)!} = \frac{n!}{(n-m)!k!(m-k)!} = \binom{n}{k}\binom{n-k}{m-k}.$$

(b) There are $\binom{n-k}{m-k}$ ways of choosing a set of m elements from $[n+1]$ that contains $n+1, n, \ldots, n-k+2$ and not $n-k+1$. Each $m+1$ subset of $[n+1]$ satisfies this condition for some k.

(c)
$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \sum_{k=0}^{m} \frac{\binom{n-k}{m-k}}{\binom{n}{m}} = \frac{\binom{n+1}{m}}{\binom{n}{m}} = \frac{n+1}{n+1-m}.$$