Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2007: Test 4

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Consider the following two one pile take-away games.
(a): In game one you can take away 1,2 or 3 chips. Prove inductively that the Grundy number $g_1(n)$ is given by $g_1(n) = n \mod 4$.
(b): In game two you can take away 1,2 or 4 chips. Prove inductively that the Grundy number $g_2(n)$ is given by $g_2(n) = n \mod 3$.
(c): Now consider the two pile game where one can either take 1,2 or 3 chips from the first pile or you can take 1,2 or 4 chips from the second pile. Is the position (150,95) a P or an N position? Justify your claim. If it is an N position, what is a correct move?
Solution:
(a) $g_1(0) = 0, g_1(1) = 1, g_1(2) = 2$ and by induction for $n \geq 3$

$$\begin{aligned} g_1(n) &= mex\{g_1(n-1), g_1(n-2), g_1(n-3)\} \\ &= mex\{n-1 \mod 4, n-2 \mod 4, n-3 \mod 4\} \\ &= n \mod 4. \end{aligned}$$

(b) $g_2(0) = 0, g_2(1) = 1, g_2(2) = 2, g_2(3) = 0$ and by induction for $n \geq 4$

$$\begin{aligned} g_2(n) &= mex\{g_2(n-1), g_2(n-2), g_2(n-4)\} \\ &= mex\{n-1 \mod 3, n-2 \mod 3, n-4 \mod 3\} \\ &= mex\{n-1 \mod 3, n-2 \mod 3\} \\ &= n \mod 3. \end{aligned}$$

(c)

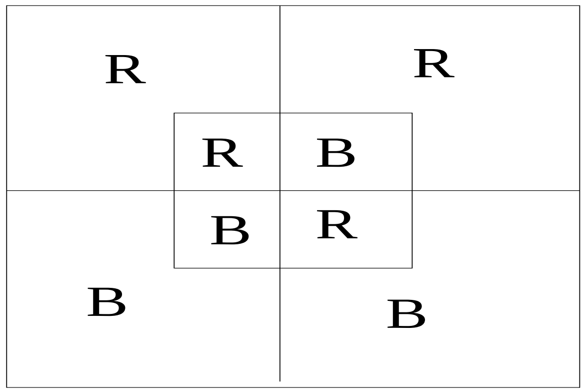
$$g(150, 95) = g_1(150) \oplus g_2(95) = 2 \oplus 2 = 0$$

and so it is a P position.

Q2: (33pts)
Consider the one pile take-away game where the set $S = \{a_1 < a_2 < \cdots < a_k = m\}$ of the possible numbers of chips to remove is finite.
(a): Show that the Grundy function g satisfies $g(n) \leq m$ for all n.
(b): Consider the intervals $I_0 = \{0, 1, \ldots, m-1\}$, $I_1 = \{m, m+1, \ldots, 2m-1\}, \ldots, I_N$ where $N = (m+1)^m$ and $I_j = \{jm, jm+1, \ldots, (j+1)m-1\}$ for $i = 0, 1, \ldots, N$. Use the pigeon-hole principle to show that there exist $s < t$ such that $g(sm+i) = g(tm+i)$ for $0 \leq i < m$.
(c): Show by induction that in fact $g(sm+i) = g(tm+i)$ for $i \geq 0$.
Solution:
(a): If $|S| \leq m$ then $mex(S) \leq m$. $g(n) = mex\{g(n-a_1), g(n-a_2), \ldots, g(n-m)\}$ and the set $|\{g(n-a_1), g(n-a_2), \ldots, g(n-m)| \leq m\}$.
(b): It follows from (a) that there are at most $(m+1)^m$ distinct possibilities for the sequence of values $g(x), x \in I_j$ for any fixed j. So, by the pigeon hole principle, there must exist $0 \leq s < t \leq N$ such that I_s and I_t generate the same sequence.
(c): With s, t as in (b) we use induction on i. It is true for $0 \leq i < m$ as this is the result of (b). Suppose that $i \geq m$. Then

$$g(sm+i) = mex\{g(sm+i-a_1), g(sm+i-a_2), \ldots, g(sm+i-a_k)\}$$
$$= mex\{g(tm+i-a_1), g(tm+i-a_2), \ldots, g(tm+i-a_k)\} = g(tm+i).$$

Q3: (34pts)
How many distinct colorings are there of the 8 region diagram below. The group G consists of 4 rotations and 4 reflections.
One possible coloring is shown.



(c): Let D denote the number of distinct colorings. We use the formula

$$D = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

g	e	a	b	c	p	q	r	s
$\lvert Fix(g) \rvert$	2^8	2^2	2^4	2^2	2^4	2^4	2^6	2^6

Here e is the identity, a, b, c are rotations and p, q, r, s are reflections.
This gives $D = 55$.