Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2007: Test 3

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Let $N = n^4 + 1$ and let $a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ be two sequences of real numbers. Show that there exist $1 \leq i_1 < i_2 \cdots < i_{n+1} \leq N$ such that **both** subsequences $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ and $b_{i_1}, b_{i_2}, \ldots, b_{i_{n+1}}$ are monotone.
Solution: The Erdős-Szekerés theorem implies that there is a monotone sub-sequence of $a_1, a_2, \ldots, a_N$ of length $n^2 + 1$. Let this sub-sequence be $a_{j_1}, a_{j_2}, \ldots, a_{j_{n^2+1}}$. Applying the Erdős-Szekerés theorem to $b_{j_1}, b_{j_2}, \ldots, b_{j_{n^2+1}}$ yields a monotone sub-sequence $b_{i_1}, b_{i_2}, \ldots, b_{i_{n+1}}$. Note that $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ is also monotone as it is a sub-sequence of $a_{j_1}, a_{j_2}, \ldots, a_{j_{n^2+1}}$.

Q2: (33pts)
Let $a_1, a_2, \ldots, a_m$ be positive integers whose sum is M. Show that if $2m > M+p$ for integer $p > 0$, then there exist $i < j$ such that $a_i+a_{i+1}+\cdots+a_j = p$. (Hint: Let $b_i = a_1+\cdots+a_i$ and consider the $2m$ integers $b_1, b_1+p, \ldots, b_m, b_m+p$).
Solution The $2m$ integers $b_1, b_1 + p, \ldots, b_m, b_m + p$ lie in $[M + p]$. By the pigeon-hole principle there must be a repeat. Now $b_j > b_i$ for $j > i$ and so there must be a $j > i$ such that $b_j = b_i+p$ and this implies $a_i+a_{i+1}+\cdots+a_j = p$.

Q3: (34pts)
(a) Show that the edges of K_5 can be colored Red and Blue so that there is no monochromatic 4-cycle.
Suppose now that the edges of K_6 are colored Red and Blue.
(b) Show that either (1) there are 3 vertices with Red degree ≥ 3 or (2) there are 3 vertices with Blue degree ≥ 3 or both (1) and (2) hold.
(c) Assume (a1) and the 3 vertices are 1,2,3 and the edge (1,2) is Blue. Show that there must be a Red 4-cycle.
(d) Assume (a1) and the 3 vertices are 1,2,3 and they form a Red triangle. Show that there is either a Red 4-cycle or a Blue 4-cycle.
Solution:
(a) Color the edges of the 5-cycle (1,2,3,4,5,1) Red and the edges of the remaining 5-cycle (1,3,5,2,4,1) Blue. There are no mono-chromatic 4-cycles.
(b) If a vertex has Red degree ≤ 2 then its Blue degree is at least 3. So if there are fewer than 3 vertices with Red degree ≥ 3, there are at least 4 vertices of Blue degree ≥ 3.
(c) Vertex 1 has ≥ 3 Red neighbours X among 3,4,5,6 and vertex 2 has ≥ 3 Red neighbours Y among 3,4,5,6. Now $|X \cap Y| = |X| + |Y| - |X \cup Y| \geq 3 + 3 - 4 = 2$. Suppose then that $3, 4$ are both Red neighbors of 1,2. Then (1,3,2,4,1) is Red.
(d) Suppose first that at least one of 4,5,6, (4 say), has 2 red neighbors (1,2 say) in 1,2,3. Then (4,1,3,2,4) is Red. Since 1,2,3 each have a red neighbor in 4,5,6, we can assume that the only Red neighbors of 1,2,3 are 4,5,6 in this order. If an edge of (4,5,6) ((4,5) say) is Red then (1,4,5,2,1) is Red. So we can assume that (4,5,6) is Blue. But we know that (3,5) and (3,6) are both Blue and so (3,5,4,6,3) is Blue.