Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2007: Test 2

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (30pts)
A Hamilton path in a tournament on vertex set $[n]$ is a permutation π of $[n]$ such that $\pi(i+1)$ beats $\pi(i)$ for $1 \leq i < n$. Suppose that $n = 2m+1$ is odd. Show that there is a tournament with $\geq m!(m+1)!/2^{n-1}$ Hamilton paths in which the odd numbered vertices are in odd position i.e. $\pi(2k+1)$ is odd for all $0 \leq k \leq m$.
Solution: There are $m!(m+1)!$ permutations π of $[n]$ in which the odd numbered vertices are in odd positions. For each there is the probability $2^{-(n-1)}$ that $\pi(i+1)$ beats $\pi(i)$ for $1 \leq i < n$. Thus the expected number of such Hamilton paths is $m!(m+1)!/2^{n-1}$ and so there must exist a tournament with at least this number.

Q2: (30pts)
Let $G = G_{n,p}$ and let $dist(x, y)$ be the minimum number of edges in a path from x to y.
($G_{n,p}$ is the graph with vertex set $[n]$ where each of the $\binom{n}{2}$ possible edges is included independently with probability p.)
Show that

$$\Pr(\exists x, y : \; dist(x, y) \geq 3) \leq \binom{n}{2}(1 - p^2)^{n-2}.$$

Solution:

$$\begin{aligned}
\Pr(\exists x, y : \; dist(x, y) \geq 3) &\leq \sum_{x,y} \Pr(dist(x, y) \geq 3) \\
&\leq \sum_{x,y} \Pr(\nexists z \neq x, y : \; (x, z, y) \textit{ is a path in } G_{n,p}) \\
&= \sum_{x,y} (1 - p^2)^{n-2} \\
&= \binom{n}{2}(1 - p^2)^{n-2}.
\end{aligned}$$

Q3: (40pts)
Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be subsets of $[n]$. Suppose that
(i) $A_i \cap B_i = \emptyset$ for $i = 1, 2, \ldots, m$.
(ii) $A_i \cap B_j \neq \emptyset$ for $i \neq j$.
Let π be a random permutation of $[n]$ and for disjoint sets A, B define the event $\mathcal{E}(A, B)$ by

$$\mathcal{E}(A, B) = \{\pi : \max\{\pi(a) : a \in A\} < \min\{\pi(b) : b \in B\}\}.$$

(a) Show that the events $\mathcal{E}_i = \mathcal{E}(A_i, B_i)$, $i = 1, 2, \ldots, m$ are disjoint.
(b) Show that for two fixed disjoint sets A, B, $|A| = a, |B| = b$ there are exactly $\binom{n}{a+b} a!b!(n-a-b)!$ permutations that produce the event $\mathcal{E}(A, B)$.
(c) Deduce that

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1. \tag{1}$$

(d) Use a suitable choice of B_i to deduce the LYM inequality from (1): The LYM inequality states that if $A_1, A_2, \ldots, A_m$ are pair-wise incomparable under set inclusion then

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1. \tag{2}$$

Solution:
(a) Suppose that $\mathcal{E}(A_i, B_i)$ and $\mathcal{E}(A_j, B_j)$ occur. Let $x \in A_i \cap B_j$ and $y \in A_j \cap B_i$. x, y exist by (ii) and (i) implies that they are distinct. Then $\mathcal{E}(A_i, B_i)$ implies that $\pi(x) < \pi(y)$ and $\mathcal{E}(A_j, B_j)$ implies that $\pi(x) > \pi(y)$, contradiction.
(b) There are $\binom{n}{a+b}$ places to position $A \cup B$. Then there are $a!b!$ that place A as the first a of these $a + b$ places. Finally, there are $(n - a - b)!$ ways of ordering the remaining elements not in $A \cup B$.
(c) Thus

$$\begin{aligned} \Pr(\mathcal{E}(A_i, B_i)) &= \frac{n!}{(|A_i| + |B_i|)!(n - |A_i| - |B_i|)!} |A_i|!|B_i|!(n - |A_i| - |B_i|)! \frac{1}{n!} \\ &= = \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}. \end{aligned}$$

But (a) implies that $\sum_{i=1}^{m} \Pr(\mathcal{E}(A_i, B_i)) \leq 1$.
(d) Put $B_i = [n] \setminus A_i$. Clearly, (i) is satisfied. Furthermore, $A_i \cap B_j = \emptyset$ iff $A_i \subseteq A_j$. So if $A_1, A_2, \ldots, A_m$ are a Sperner family, (b) holds. Inequality (2) follows from (1) and $|A_i| + |B_i| = n$.