Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2007: Test 1

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Show that the number of ways of placing k 1's and $n-k$ 0's on the vertices of an n vertex polygon so that each 1 is seperated by at least 2 0's is $\frac{n}{k}\binom{n-2k-1}{k-1}$.
Solution Let the 1's be at position $x_1, x_2, \ldots, x_k$ in order round the polygon. There are n choices for x_1 and then if y_i is the number of zero's between x_i and x_{i+1} for $i = 1, 2, \ldots, k$ (here $x_{k+1} = x_1$) then (i) $y_i \geq 2$, $i = 1, 2, \ldots, k$ and (ii) $y_1 + \ldots + y_k = n - k$. There are $\binom{n-2k-1}{k-1}$ choices for the y's and n choices for x_1. Each placement arises exactly k times in this manner.

Q2: (33pts)
(a): We have n boxes $B_1, B_2, \ldots, B_n$ and kn distinguishable balls $b_1, b_2, \ldots, b_{kn}$. Show that there are $\frac{(kn)!}{k!^n}$ ways to place the balls into the boxes so that each box gets k balls.
(b): An allocation of balls to boxes is said to be *scrambled* if there does **not** exist i such that box B_i contains balls $b_{(i-1)k+1} \ldots, , b_{ik}$. Use the Inclusion-Exclusion formula to determine the number of scrambled allocations.
Re-call that if $A_1, A_2, \ldots, A_N \subseteq A$ then

$$\left| \bigcap_{i=1}^{N} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Solution
(a) An allocation of balls to boxes gives rise to a permutation of the balls, where the balls in box B_i precede the balls in box B_{i+1} for $1 \leq i < n$. There are $(kn)!$ permutations and each allocation gives rise to $(k!)^n$ permutations, since the balls within a box can permuted arbitrarily.
(b) Let $A_i = \{\textit{allocations in which box } B_i \textit{ gets balls } b_{(i-1)k+1}, \ldots, b_{ik}\}$. We want $\left|\bigcap_{i=1}^{N} \bar{A}_i\right|$. From (a) we get $|A_S| = \frac{(k(n-|S|))!}{(k!)^{n-|S|}}$ and then the inclusion-exclusion formula gives

$$\left| \bigcap_{i=1}^{N} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} \frac{(k(n-|S|))!}{(k!)^{n-|S|}} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(k(n-i))!}{(k!)^{n-i}}.$$

Q3: (34pts) The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1$ and

$$a_n - 4a_{n-1} = 4^n$$

for $n \geq 1$.
(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
(b): Find an expression for a_n, $n \geq 0$.
Solution
Multiply each equation by x^n and sum. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 4a_{n-1})x^n &= \sum_{n=1}^{\infty} 4^n x^n. \\
(a(x) - 1) - 4xa(x) &= \frac{1}{1-4x} - 1. \\
a(x) &= \frac{1}{(1-4x)^2} \\
&= \sum_{n=0}^{\infty}(n+1)4^n x^n.
\end{aligned}$$

So,

$$a_n = (n+1)4^n.$$